

Danby's Solution of Kepler's Equation

This program (demokep1) demonstrates how to interact with the *Numerit* function `kepler1` which solves the elliptic and hyperbolic form of Kepler's equation using a numerical method devised by Professor J.M.A. (Tony) Danby at North Carolina State University. Additional information about this algorithm can be found in the technical papers "The Solution of Kepler's Equation", *Celestial Mechanics*, **31** (1983) 95-107, 317-328 and **40** (1987) 303-312.

The *elliptic* form of Kepler's equation is given by the expression

$$M = E - e \sin E \quad (1)$$

where M is the mean anomaly, E is the eccentric anomaly and e is the orbital eccentricity.

The eccentric anomaly *initial guess* for Danby's method is

$$E_0 = M + 0.85 \operatorname{sign}(\sin M) e \quad (2)$$

The *fundamental equation* we want to solve is

$$f(E) = E - e \sin E - M = 0 \quad (3)$$

which has the first three derivatives given by

$$f'(E) = 1 - e \cos E$$

$$f''(E) = e \sin E \quad (4)$$

$$f'''(E) = e \cos E$$

The iteration for an *updated* eccentric anomaly E_{n+1} based on a current value E_n is given by the next four equations:

$$\begin{aligned} \Delta(E_n) &= -\frac{f}{f'} \\ \Delta^*(E_n) &= -\frac{f}{f' + \frac{1}{2}\Delta f''} \\ \Delta_n(E_n) &= -\frac{f}{f' + \frac{1}{2}\Delta f'' + \frac{1}{6}\Delta^2 f'''} \end{aligned} \quad (5)$$

$$E_{n+1} = E_n + \Delta_n$$

Celestial Computing with Numerit

This algorithm provides *quartic* convergence of Kepler's equation. The process summarized by Eq. 5 is repeated until the following convergence test involving the fundamental equation is satisfied:

$$|f(E)| \leq \mathbf{e} \quad (6)$$

where \mathbf{e} is the convergence tolerance. This tolerance is "hardwired" in the software to the value `ktol = 1.0e-10`.

Finally, the true anomaly can be calculated with the following two equations

$$\sin \mathbf{n} = \sqrt{1 - e^2} \sin E \quad (7)$$

$$\cos \mathbf{n} = \cos E - e$$

and the four quadrant inverse tangent given by

$$\mathbf{n} = \arctan(\sin \mathbf{n}, \cos \mathbf{n}) \quad (8)$$

If the orbit is hyperbolic the initial guess is

$$H_0 = \log\left(\frac{2M}{e} + 1.8\right) \quad (9)$$

where H_0 is the hyperbolic anomaly.

The fundamental equation and first three derivatives for this case are as follows:

$$f(H) = e \sinh H - H - M$$

$$f'(H) = e \cosh H - 1 \quad (10)$$

$$f''(H) = e \sinh H$$

$$f'''(H) = e \cosh H$$

Otherwise, the iteration loop which calculates Δ , Δ^* and so forth is the same. The true anomaly for hyperbolic orbits is determined with the next two equations and a four quadrant inverse tangent function as before.

$$\sin \mathbf{n} = \sqrt{e^2 - 1} \sinh H \quad (11)$$

$$\cos \mathbf{n} = e - \cosh H$$

Celestial Computing with Numerit

The syntax for this *Numerit* function is as follows:

```
function kepler1 (manom, ecc, eanom, tanom)

` solve Kepler's equation for circular,
` elliptic and hyperbolic orbits

` Danby's method

` input

` manom = mean anomaly (radians)
` ecc   = orbital eccentricity (non-dimensional)

` output

` eanom = eccentric anomaly (radians)
` tanom = true anomaly (radians)
```

The program will interactively prompt you for the orbital eccentricity. It will then calculate and plot arrays containing the number of algorithm iterations and the algorithm *absolute* error (Equation 3).

The following are plots of the number of algorithm iterations and the absolute error for an orbital eccentricity of 0.735. This is the eccentricity of a typical Molniya orbit. For this example Danby's method requires at most 3 iterations and the absolute value of the algorithm error is always less than 10^{-12} .

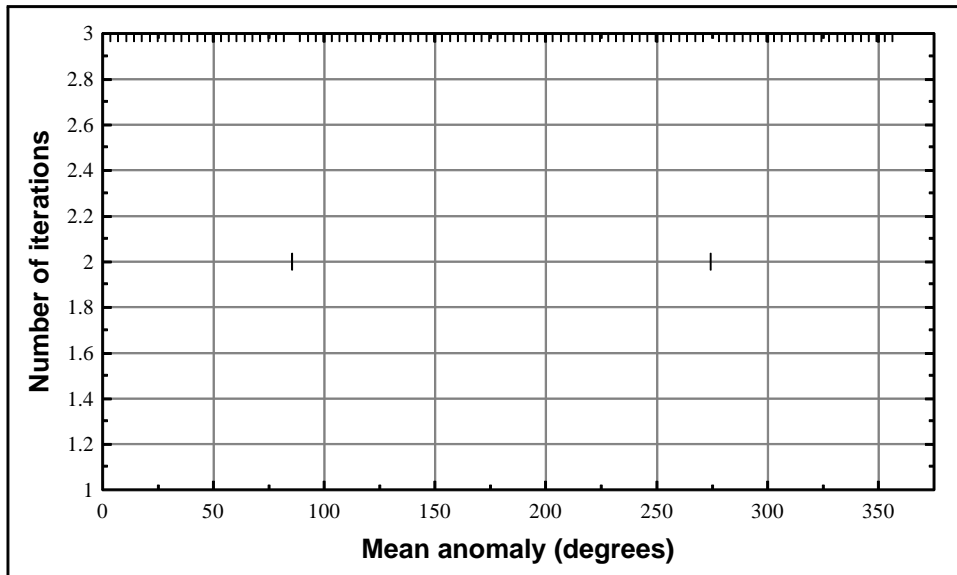


Figure 1. Algorithm iterations

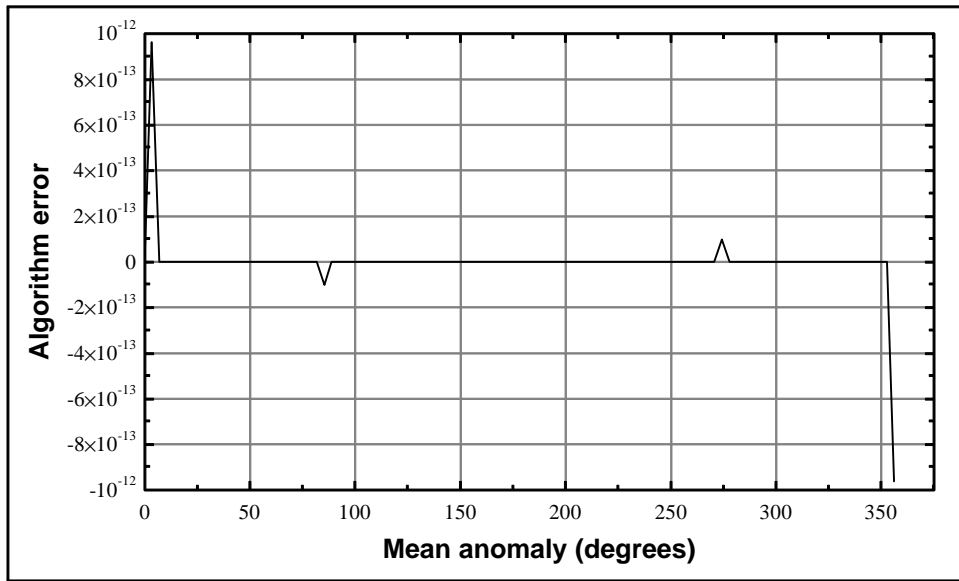


Figure 2. Algorithm error