The Circular Restricted Three-body Problem

This ***Numerit*** program (`g3body`) graphically displays motion in the circular-restricted three-body problem (CRTBP). This algorithm and program examples are based on the methods described in "Periodic Orbits in the Restricted Three-Body Problem with Earth-Moon Masses", by R. A. Broucke, JPL TR 32-1168, 1968. In the discussion which follows, subscript 1 is the Earth and subscript 2 is the Moon.

The system of second order vector differential equations of motion of a point mass satellite in the CRTBP are given by

$$\begin{aligned} \frac{d^2x}{dt^2} - 2\frac{dy}{dt} - x &= -(1-m)\frac{x-x_1}{r_1^3} - m\frac{x-x_2}{x_2^3} \\ \frac{d^2y}{dt^2} - 2\frac{dx}{dt} - y &= -(1-m)\frac{y}{r_1^3} - m\frac{y}{x_2^3} \end{aligned} \tag{1}$$

where

$x = x$ component of position

$y = y$ component of position

$x_1 = -m$

$x_2 = 1 - m$

$m =$ Earth-Moon mass ratio $= m_1 / m_2 \approx 1/81.27$

$r_1^2 = (x - x_1)^2 + y^2$

$r_2^2 = (x - x_2)^2 + y^2$

The motion of the spacecraft is displayed in a coordinate system which is rotating about the center-of-mass or barycenter of the Earth-Moon system. The motion is confined to the x-y plane.

For convenience, the problem is formulated in canonical units. The unit of length is taken to be the constant distance between the Earth and Moon, and the unit of time is chosen such that the Earth and Moon have an angular velocity w about the barycenter equal to 1. Kepler's third law is then

$$w^2 \left| m_1 m_2 \right|^3 = g(m_1 + m_2) = 1 \tag{2}$$

In his technical report, Professor Brouke calls these *synodical* coordinates.

In his prize memoir of 1772, Joseph-Louis Lagrange proved that there are five *equilibrium points* in the circular-restricted three-body problem. If we place a satellite or celestial body at one of these points with zero initial velocity, it will stay there permanently. These special positions are also called *libration* points.

For the Earth-Moon mass ratio value of , the x and y coordinates of these five equilibrium points L_i are as follows:

L_1 `x = +0.836892919    y = 0`
L_2 `x = +1.155699520    y = 0`
L_3 `x = -1.005064520    y = 0`
L_4 `x = +0.487844901    y = +0.866025404`
L_5 `x = +0.487844901    y = +0.866025404`

Three of these libration points are on the x-axis and the other two form equilateral triangles with the positions of the Earth and Moon.

The program begins by asking you to select from 4 main program options. The first three options will display typical periodic orbits about the respective libration point. The initial conditions for each of these orbits are as follows:

(1) periodic orbit about the L_1 libration point

$$\begin{aligned} x_0 &= 0.300000161 \\ \dot{y}_0 &= -2.536145497 \\ m &= 0.012155092 \\ t_f &= 5.349501906 \end{aligned}$$

(2) periodic orbit about the L_2 libration point

$$\begin{aligned} x_0 &= 2.840829343 \\ \dot{y}_0 &= -2.747640074 \\ m &= 0.012155085 \\ t_f &= 11.933318588 \end{aligned}$$

(3) periodic orbit about the L_3 libration point

$$\begin{aligned} x_0 &= -1.600000312 \\ \dot{y}_0 &= 2.066174572 \\ m &= 0.012155092 \\ t_f &= 6.303856312 \end{aligned}$$

Notice that each orbit is defined by an initial x-component of position, x_0 , an initial y-component of velocity, $\dot{y}_0$, a value of Earth-Moon mass ratio m , and an orbital period or final time t_f . The initial y-component of position and x-component of velocity for each these orbits is equal to zero. The software will draw the location of the libration point on the graphics display.

If you would like to experiment with your own initial conditions, select option 4. The program will then prompt you for the initial x and y components of the orbit's position and velocity vector, and the value of to use in the simulation. The software will also ask for an initial and final time to run the simulation, and an integration step size. The value 0.01 is a good number for step size.

Here is a short list of initial conditions for several other periodic orbits which you may want to display with this program.

(1) Retrograde periodic orbit about m_1

$$\begin{aligned} x_0 &= -2.499999883 \\ \dot{y}_0 &= 2.100046263 \\ m &= 0.012155092 \\ t_f &= 11.99941766 \end{aligned}$$

(2) Direct periodic orbit about m_1

$$\begin{aligned} x_0 &= 0.952281734 \\ \dot{y}_0 &= -0.957747254 \\ m &= 0.012155092 \\ t_f &= 6.450768946 \end{aligned}$$

(3) Direct periodic orbit about m_1 and m_2

$$\begin{aligned} x_0 &= 3.147603117 \\ \dot{y}_0 &= -3.07676285 \\ m &= 0.012155092 \\ t_f &= 12.567475674 \end{aligned}$$

(4) Direct periodic orbit about m_2

$$\begin{aligned} x_0 &= 1.399999991 \\ \dot{y}_0 &= -0.9298385561 \\ m &= 0.012155092 \\ t_f &= 13.775148738 \end{aligned}$$

The following is a plot of a periodic orbit about the L_2 libration point. This plot was created by selecting program option 2.

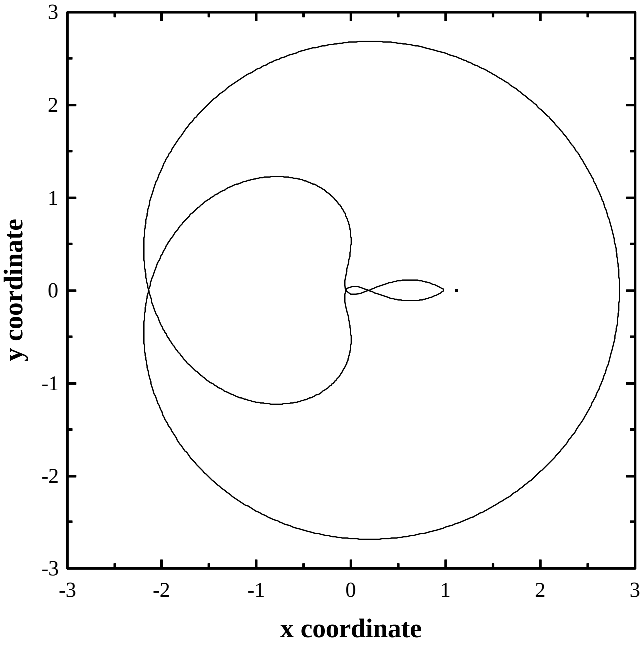

Figure 1. Periodic Orbit about the L_2 Libration Point

For a correct graphic display, the *Width* and *Height* dimensions should be the same as well as the x and y axes scales. You can change these properties interactively by double clicking on this graph with the left mouse button.