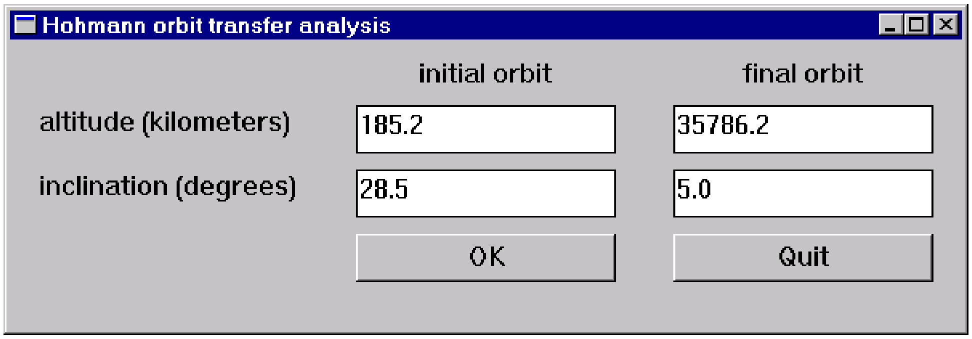
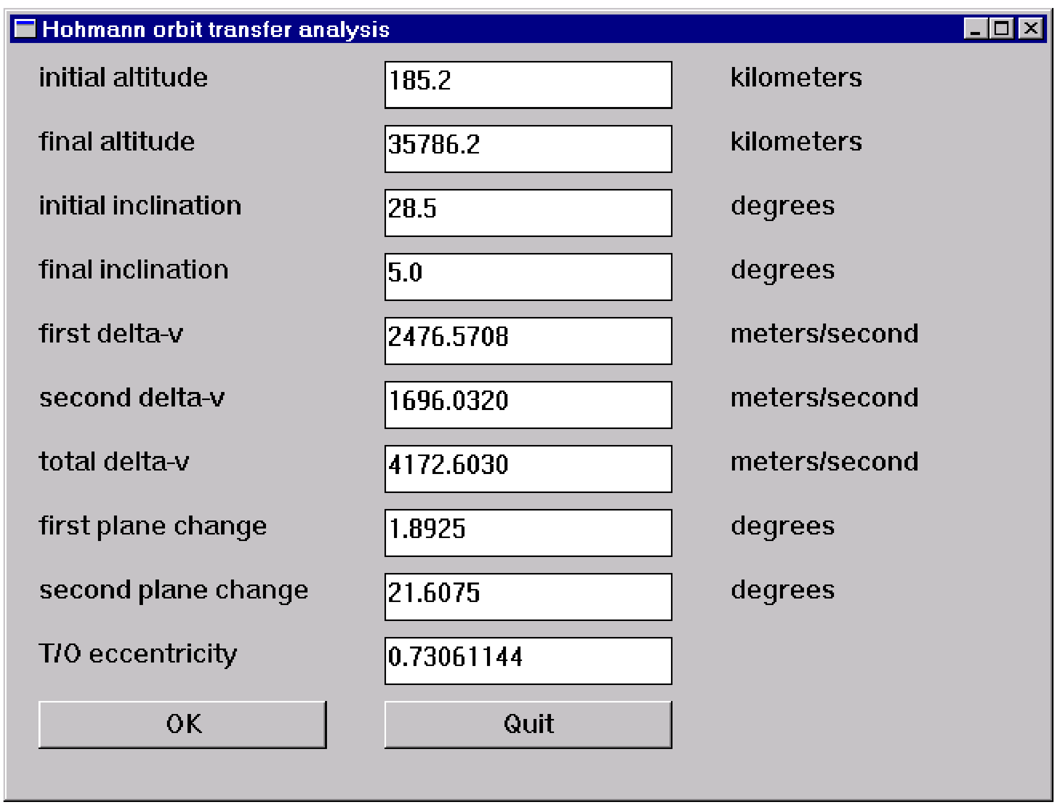
The Hohmann Orbit Transfer

This Windows XP/Vista application (`hohmann.exe`) can be used to determine optimum two impulse Hohmann orbital transfers between coplanar and non-coplanar circular orbits. This computer program is written in Visual Fortran and uses the DISLIN graphics library to display three-dimensional graphics of the orbital transfer.

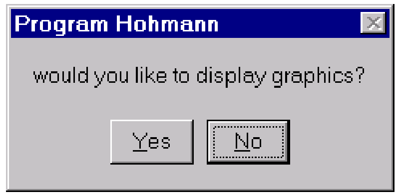
The altitudes and orbital inclinations of the initial and final orbits are input to the program using the following main screen:



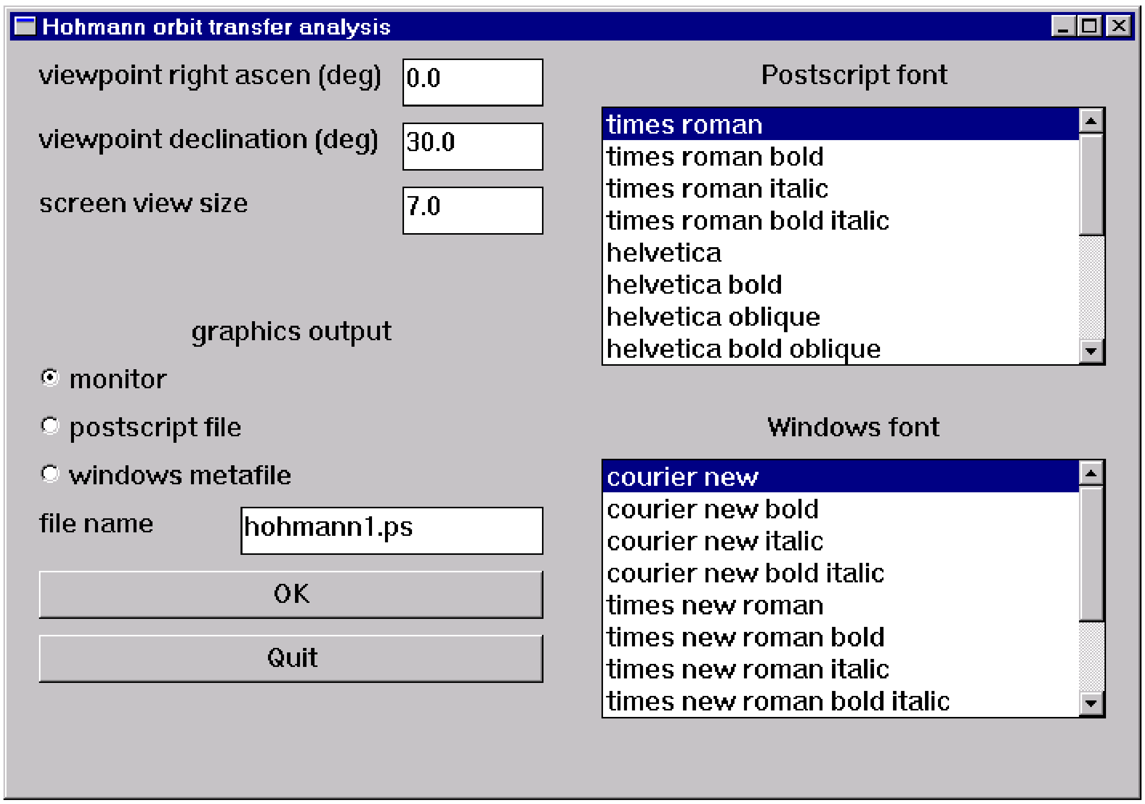
The software will then display a data screen which summarizes the characteristics of the orbital transfer. The following is the program output for this example.



After the data display screen is created and displayed, the program will ask if you would like to create a three-dimensional display of the orbit transfer with

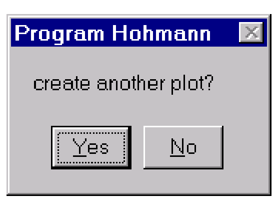


If the user clicks on the Yes button, the program will display the following *graphics setup* screen.

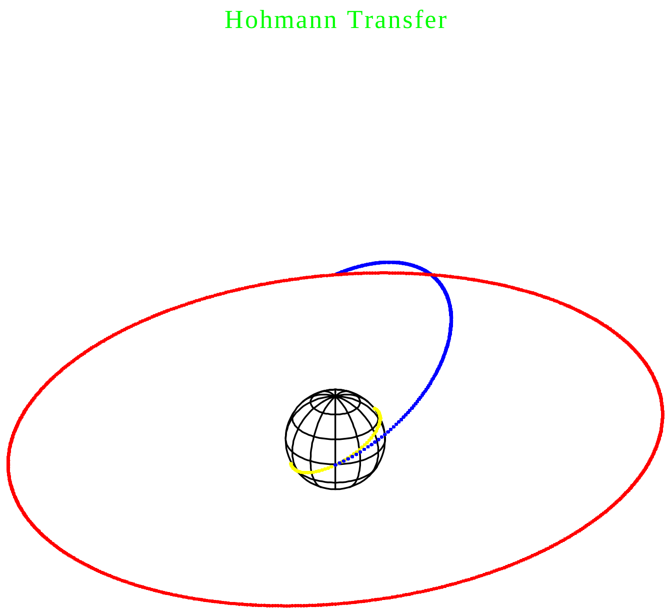


This screen allows the user to define the graphics viewpoint as well as the size of the view. You can also specify the font to use and the destination of the graphics image. Please note that the Windows font is valid for both monitor and windows metafile graphics.

After the graphics is displayed and the user continues the program by pressing the right mouse button, the software will ask if you would like to create another plot with the following prompt:



The following is a typical graphics display for this non-coplanar Hohmann orbit transfer.



In this illustration, the initial orbit is the yellow trace, the transfer orbit is blue and the final orbit is the red trace.

Technical Discussion

The coplanar circular orbit-to-circular orbit transfer was discovered by the German engineer Walter Hohmann in 1925. The transfer consists of a velocity impulse on an initial circular orbit, in the direction of motion and collinear with the velocity vector, which propels the space vehicle into an elliptical transfer orbit. At a transfer angle of 180 degrees from the first impulse, a second velocity impulse or ΔV, also collinear and in the direction of motion, places the vehicle into a final circular orbit at the desired final altitude. The impulsive ΔV assumption means that the velocity, but not the position, of the vehicle is changed instantaneously. This is equivalent to a rocket engine with infinite thrust magnitude. Therefore, the Hohmann formulation is the ideal and minimum energy solution to this type of orbit transfer problem.

Coplanar Equations

For the coplanar Hohmann transfer both velocity impulses are confined to the orbital planes of the initial and final orbits. The first impulse creates an elliptical transfer orbit with a perigee altitude equal to the altitude of the initial circular orbit and an apogee altitude equal to the altitude of the final orbit. The second impulse circularizes the transfer orbit at apogee. Both impulses are *posigrade* which means that they are in the direction of orbital motion.

We begin by defining three *normalized* radii as follows:

$$R_1 = \sqrt{2\frac{r_f}{r_i + r_f}}$$

$$R_2 = \sqrt{\frac{r_i}{r_f}} \qquad (1)$$

$$R_3 = \sqrt{2\frac{r_i}{r_i + r_f}}$$

where r_i is the geocentric radius of the initial circular park orbit and r_f is the radius of the final circular mission orbit. The relationship between radius and initial orbit altitude h_i and the final orbit altitude h_f is as follows:

$$r_i = r_e + h_i$$

$$r_f = r_e + h_f \qquad (2)$$

where r_e is the radius of the Earth.

The magnitude of the first impulse is

$$\Delta V_1 = V_{lc}\sqrt{1 + R_1^2 - 2R_1} \qquad (3)$$

and is simply the difference between the speed on the initial orbit and the perigee speed of the transfer orbit. The scalar magnitude of the second impulse is

$$\Delta V_2 = V_{lc}\sqrt{R_2^2 + R_2^2 R_3^2 - 2R_2^2 R_3} \qquad (4)$$

which is the difference between the speed on the final orbit and the apogee speed of the transfer ellipse. In each of these ΔV equations V_{lc} is called the *local circular velocity*. It can be determined from

$$V_{lc} = \sqrt{\frac{\mu}{r_i}} \qquad (5)$$

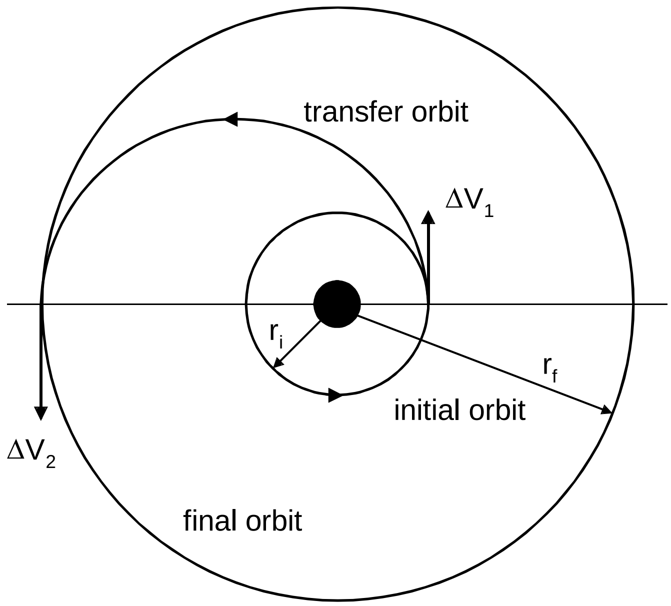
and represents the scalar speed in the initial orbit. In these equations μ is the gravitational constant of the central body. The transfer time from the first impulse to the second is equal to one half the orbital period of the transfer ellipse

$$\tau = \pi\sqrt{\frac{a^3}{\mu}} \qquad (6)$$

where a is the semimajor axis of the transfer orbit and is equal to $\left(r_i + r_f\right)/2$.
The orbital eccentricity of the transfer ellipse is

$$e = \frac{\max\left(r_i, r_f\right) - \min\left(r_i, r_f\right)}{r_f - r_i} \tag{7}$$

The following diagram illustrates the geometry of the coplanar Hohmann transfer.



Non-coplanar Equations

The non-coplanar Hohmann transfer involves orbital transfer between two circular orbits which have different orbital inclinations. For this problem the propulsive energy is minimized if we optimally partition the total orbital inclination change between the first and second impulses.

The scalar magnitude of the first impulse is

$$\Delta V_1 = V_{lc}\sqrt{1 + R_1^2 - 2R_1\cos\theta_1} \tag{8}$$

where θ_1 is the plane change associated with the first impulse. The magnitude of the second impulse is

$$\Delta V_2 = V_{lc}\sqrt{R_2^2 + R_2^2 R_3^2 - 2R_2^2 R_3 \cos\theta_2} \tag{9}$$

where θ_2 is the plane change associated with the second impulse. These two equations are different forms of the law of cosines.

The total ΔV required for the maneuver is the sum of the two impulses as follows

$$\Delta V = \Delta V_1 + \Delta V_2 \tag{10}$$

The relationship between the plane change angles is $\theta_t = \theta_1 + \theta_2$ where θ_t is the total plane change angle between the initial and final orbits.

Optimizing the non-coplanar Hohmann transfer involves allocating the total plane change angle between the two maneuvers such that the total ΔV required for the mission is minimized. We can determine this answer by solving for the root of a derivative.

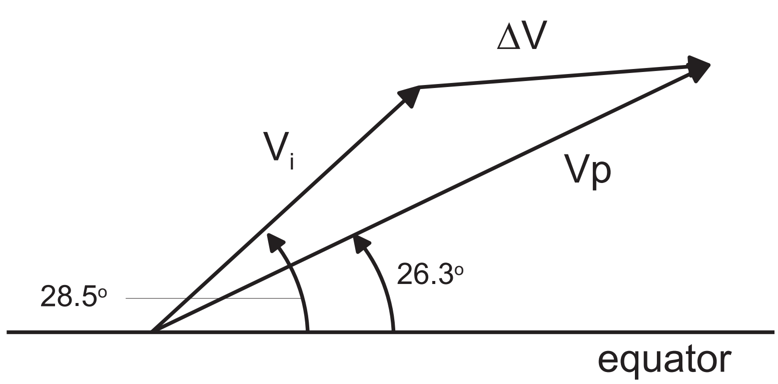
The partial derivative of the total ΔV with respect to the first plane change angle is given by:

$$\frac{\partial \Delta V}{\partial \theta_1} = \frac{R_1 \sin\theta_1}{\sqrt{1 + R_1^2 - 2R_1\cos\theta_1}} + \frac{R_2^2 R_3 \left(\sin\theta_t \cos\theta_1 - \cos\theta_t \sin\theta_1\right)}{\sqrt{R_2^2 + R_2^2 R_3^2 - 2R_2^2 R_3 \cos\left(\theta_t - \theta_1\right)}} \tag{11}$$

If we determine the value of θ_1 which makes this derivative zero, we have found the optimum plane change required at the first impulse. Once θ_1 is calculated, we can determine θ_2 from the total plane change angle relationship and the velocity impulses from the equations 8 and 9.

To illustrate the geometry of a non-coplanar Hohmann transfer, let's look at a typical orbit transfer from a low altitude Earth orbit (LEO) at an altitude of 185.2 kilometers and an orbital inclination of 28.5 degrees to a geosynchronous Earth orbit (GEO) at an altitude of 35786.36 kilometers and 0 degrees inclination.

The following is a ΔV diagram for the first maneuver of this orbit transfer example. In this view we are looking along the line of nodes which is the mutual intersection of the park and transfer orbit planes with the equatorial plane.



In this diagram V_i is the speed on the initial park orbit, V_p is the perigee speed of the elliptic transfer orbit, and ΔV is the ΔV required for the first maneuver. The inclinations of the park and transfer orbit are also labeled. From this geometry and the law of cosines, the required ΔV is given by

$$\Delta V = \sqrt{V_i^2 + V_p^2 - 2V_i V_p \cos\Delta i} \tag{12}$$

where Δi is the inclination difference or plane change between the park and transfer orbits.