Gravity-Assist Trajectory Design

This interactive Windows XP/Vista computer program (`flyby.exe`) determines the characteristics of *patched-conic* heliocentric orbital transfers between any two planets of our solar system with an intermediate gravity-assist flyby of a third planet. The software implements a flyby and arrival calendar date grid search technique.

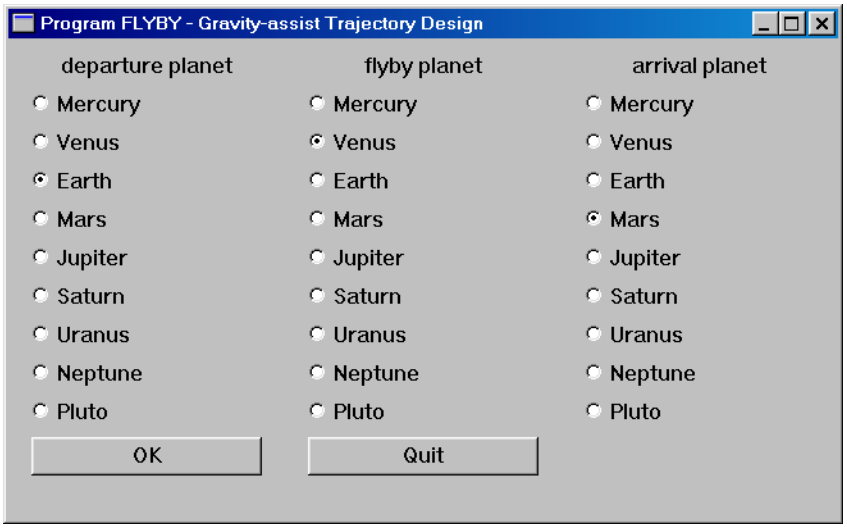
This application provides the following information:

- launch energy and calendar dates and times of the flyby and arrival
- classical orbital elements of each segment of the trajectory
- orbital elements of the flyby hyperbola
- graphics display of the heliocentric trajectory and planetary orbits

The inputs required by this computer program are minimal. The user provides the following information:

(1) departure calendar date and universal time
(2) departure, flyby and arrival planets
(3) desired flyby altitude
(4) size and number of time steps for each leg of the heliocentric trajectory

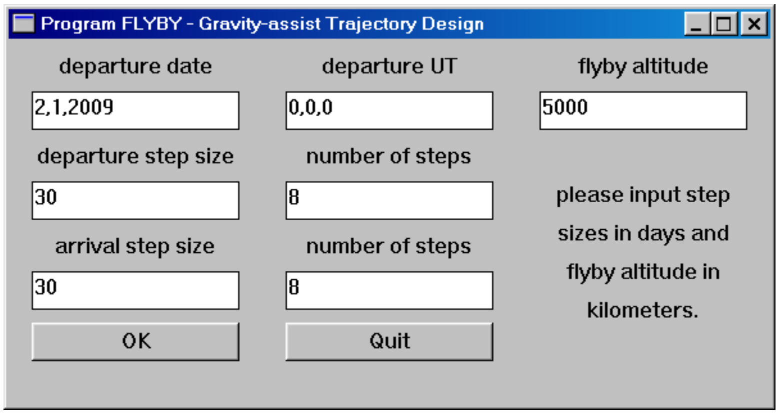
The software begins by displaying the following *planet selection screen*:



departure planet	flyby planet	arrival planet
○ Mercury	○ Mercury	○ Mercury
○ Venus	● Venus	○ Venus
● Earth	○ Earth	○ Earth
○ Mars	○ Mars	● Mars
○ Jupiter	○ Jupiter	○ Jupiter
○ Saturn	○ Saturn	○ Saturn
○ Uranus	○ Uranus	○ Uranus
○ Neptune	○ Neptune	○ Neptune
○ Pluto	○ Pluto	○ Pluto

This screen allows the user to select the departure, flyby and arrival planets.

The next screen displayed by the software allows the user to specify the launch calendar date and universal time, the flyby altitude and the search parameters.


Program FLYBY - Gravity-assist Trajectory Design

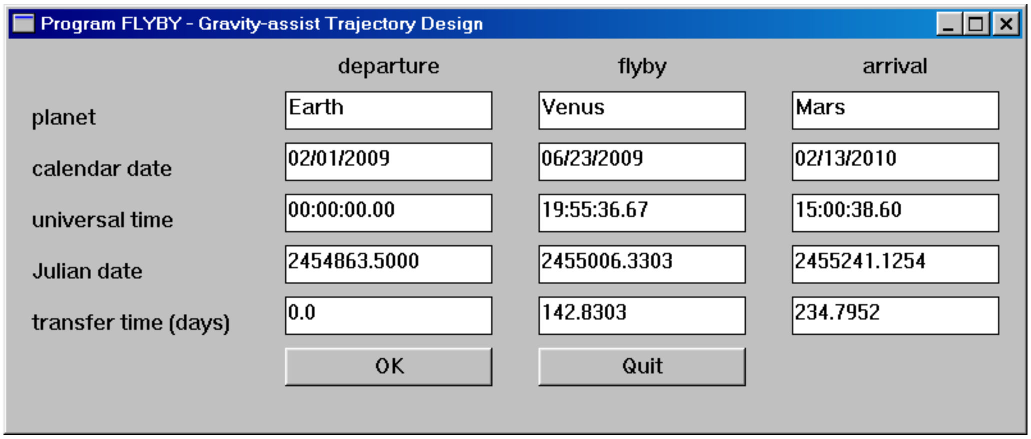
departure date: 2,1,2009
departure UT: 0,0,0
flyby altitude: 5000
departure step size: 30
number of steps: 8
arrival step size: 30
number of steps: 8

please input step sizes in days and flyby altitude in kilometers.

During the search the software will display the message

```
please wait, searching for a solution ...
```

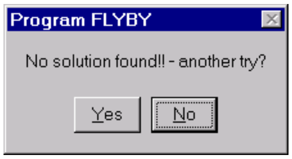
After a valid solution is found the program will display a date and universal time summary similar to the following:


Program FLYBY - Gravity-assist Trajectory Design

	departure	flyby	arrival
planet	Earth	Venus	Mars
calendar date	02/01/2009	06/23/2009	02/13/2010
universal time	00:00:00.00	19:55:36.67	15:00:38.60
Julian date	2454863.5000	2455006.3303	2455241.1254
transfer time (days)	0.0	142.8303	234.7952

The transfer time displayed in the "flyby" column is the duration of the first leg (from departure planet to flyby planet) of the transfer orbit. The transfer time displayed in the "arrival" column is the duration of the second leg (from flyby planet to arrival planet) of the interplanetary transfer orbit.

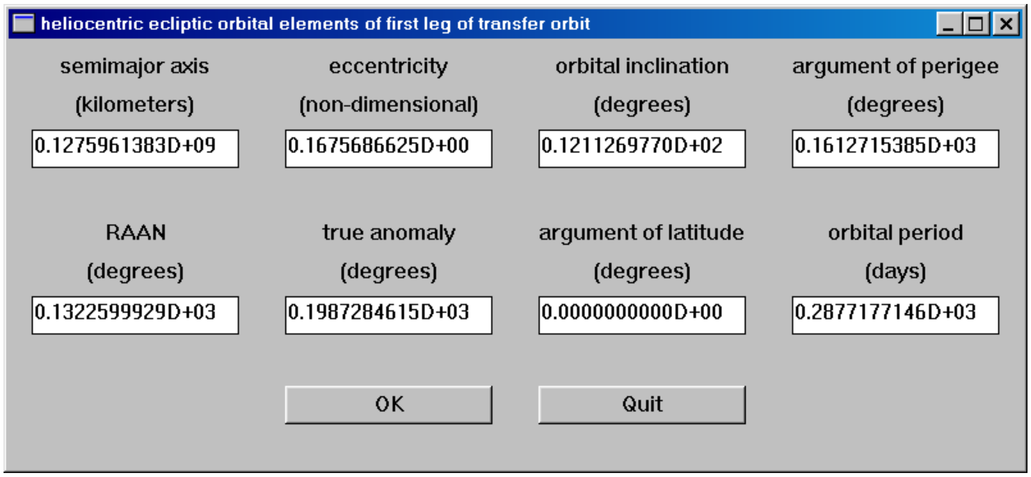
If a solution is not found, the software will display the following screen:


Program FLYBY

No solution found!! - another try?

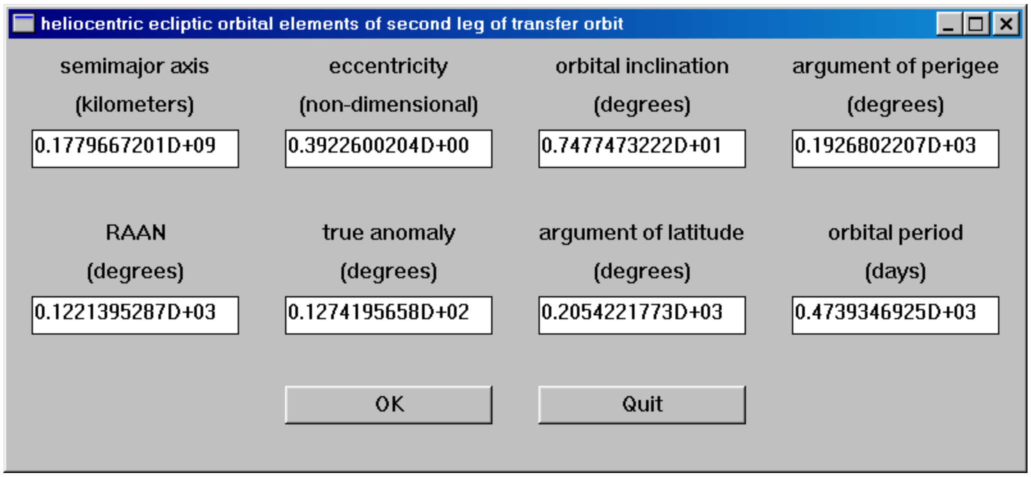
A user response of Yes will return to the planet selection screen.

The software will then display the heliocentric ecliptic classical orbital elements of the first leg of the transfer orbit. The following is the screen for this example.


heliocentric ecliptic orbital elements of first leg of transfer orbit

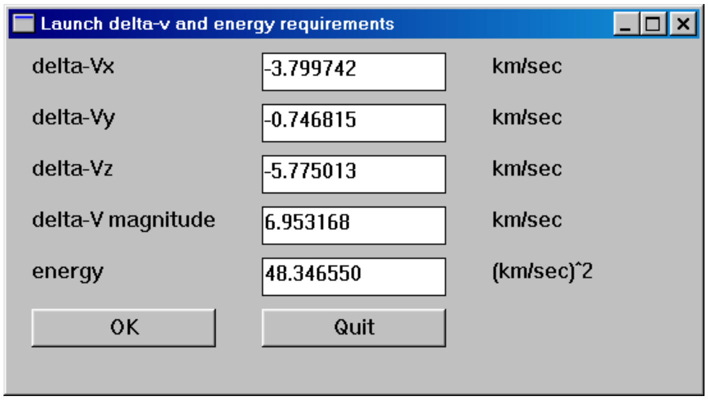
semimajor axis (kilometers): 0.1275961383D+09
eccentricity (non-dimensional): 0.1675686625D+00
orbital inclination (degrees): 0.1211269770D+02
argument of perigee (degrees): 0.1612715385D+03
RAAN (degrees): 0.1322599929D+03
true anomaly (degrees): 0.1987284615D+03
argument of latitude (degrees): 0.0000000000D+00
orbital period (days): 0.2877177146D+03

The software will then display the heliocentric ecliptic classical orbital elements of the second leg of the transfer orbit. The following is the screen for this example.


heliocentric ecliptic orbital elements of second leg of transfer orbit

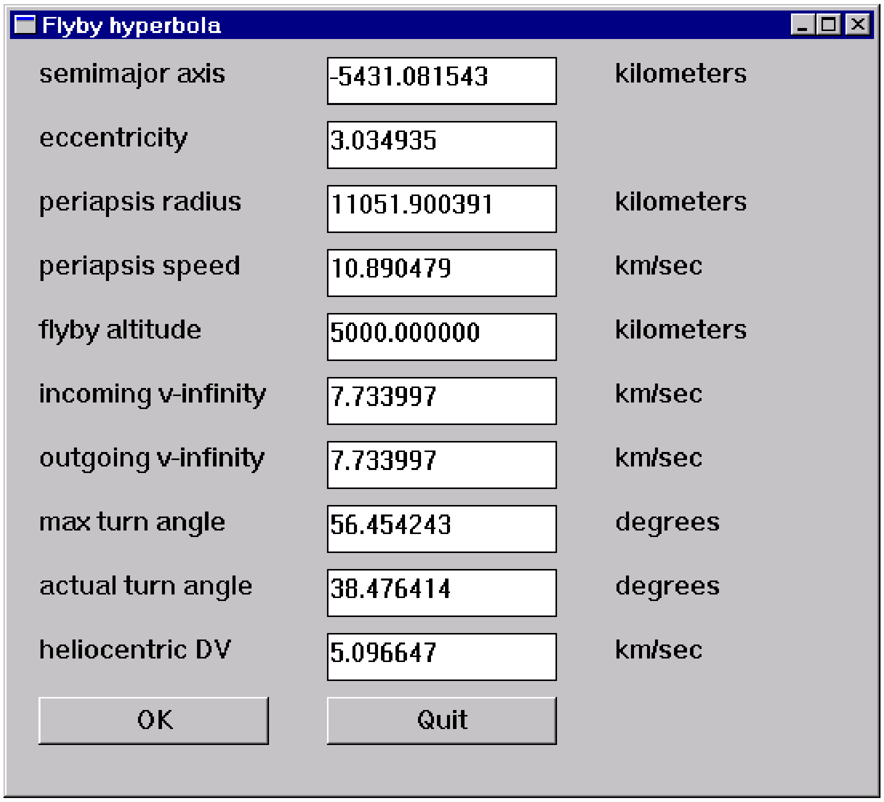
semimajor axis (kilometers): 0.1779667201D+09
eccentricity (non-dimensional): 0.3922600204D+00
orbital inclination (degrees): 0.7477473222D+01
argument of perigee (degrees): 0.1926802207D+03
RAAN (degrees): 0.1221395287D+03
true anomaly (degrees): 0.1274195658D+02
argument of latitude (degrees): 0.2054221773D+03
orbital period (days): 0.4739346925D+03

The computer program will display the launch delta-v and energy. For this example the screen display is


Launch delta-v and energy requirements

delta-Vx: -3.799742 km/sec
delta-Vy: -0.746815 km/sec
delta-Vz: -5.775013 km/sec
delta-V magnitude: 6.953168 km/sec
energy: 48.346550 (km/sec)^2

Finally, the software will also display the characteristics of the flyby hyperbola. This screen is as follows:



semimajor axis -5431.081543 kilometers
eccentricity 3.034935
periapsis radius 11051.900391 kilometers
periapsis speed 10.890479 km/sec
flyby altitude 5000.000000 kilometers
incoming v-infinity 7.733997 km/sec
outgoing v-infinity 7.733997 km/sec
max turn angle 56.454243 degrees
actual turn angle 38.476414 degrees
heliocentric DV 5.096647 km/sec

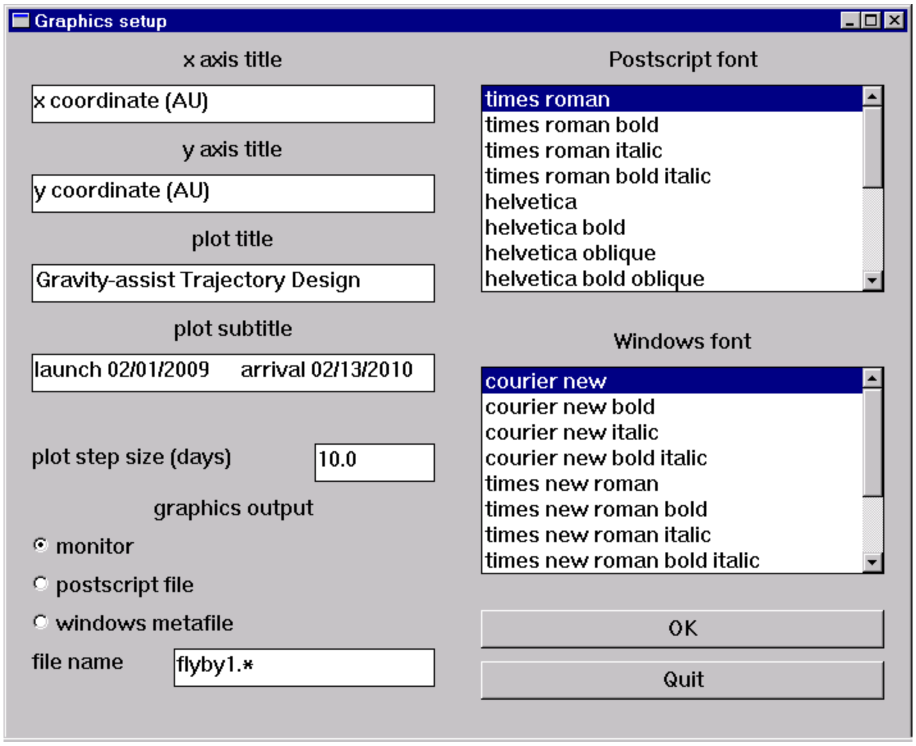
The heliocentric DV is the scalar speed increase, relative to the Sun, due to the flyby. The flyby altitude, incoming v-infinity and outgoing v-infinity are the actual "solved-for" values. These values indicate how well the numerical method actually solved this trajectory optimization problem.

Trajectory Graphics

After displaying all numerical data, the computer program will ask if you would like to create a graphics display of the gravity-assist trajectory. The screen prompt for this program option is

would you like to display graphics?

A user response of No will return to the planet selection screen. A user response of Yes will display the following *graphics setup screen*.



x axis title: x coordinate (AU)

y axis title: y coordinate (AU)

plot title: Gravity-assist Trajectory Design

plot subtitle: launch 02/01/2009 arrival 02/13/2010

plot step size (days): 10.0

graphics output

● monitor

○ postscript file

○ windows metafile

file name: flyby1.*

This screen allows the user to create custom axes and plot labels as well as the destination of the graphics image. The software will automatically create a plot subtitle consisting of the calendar dates of launch and arrival. You can also specify the font to use and the step size for the trajectory plot. Please note that the Windows font is valid for both monitor and Windows metafile graphics. A plot step size between 5 and 10 days is recommended.

The following is a typical graphic display of the heliocentric transfer trajectory for this example. In this plot we are looking down on the ecliptic plane from the north celestial pole. The launch planet is labeled with an L, the flyby planet is labeled with an F and the arrival planet is labeled with an A. The planet orbits and transfer trajectory are labeled with tick marks every 10 days.

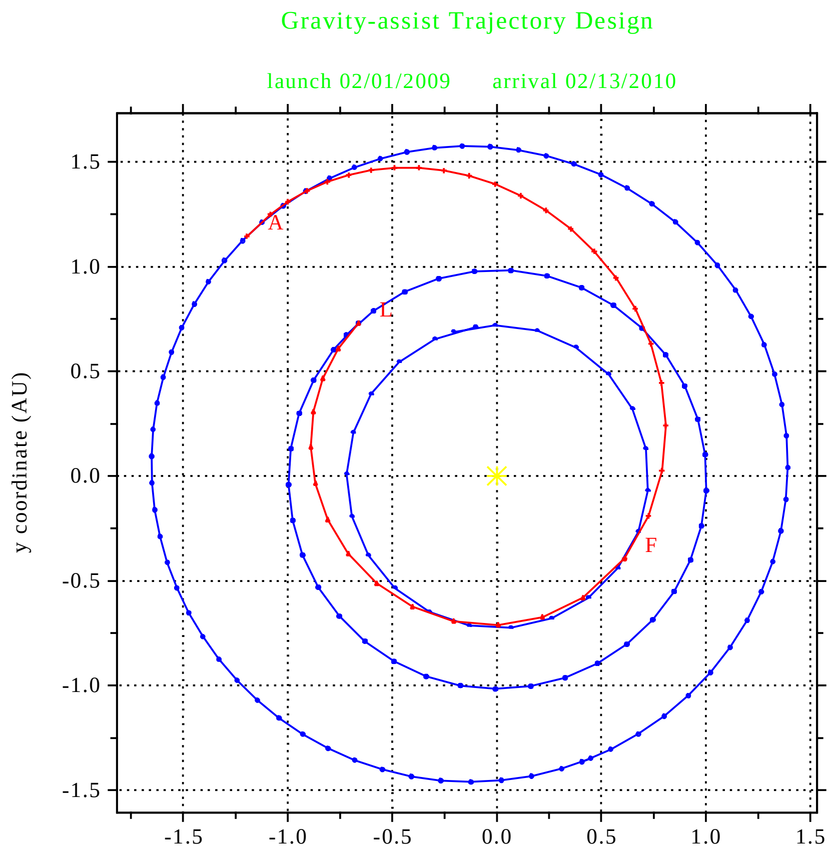


Figure 1 Heliocentric Transfer Trajectory

Technical Discussion

A patched-conic heliocentric transfer trajectory is represented in this computer program by the following sequential trajectory segments:

(1) heliocentric elliptical orbit from the departure planet to the flyby planet

(2) planetocentric hyperbola relative to the flyby planet

(3) heliocentric ellipse from the flyby planet to the destination planet

Each trajectory segment is modeled as two-body motion governed by the respective central body. During the heliocentric cruise segments, the Sun is the central body and during the gravity-assist encounter the flyby planet is the central body.

The vector relationships between the incoming v-infinity vector $\mathbf{v}_\infty^-$, the outgoing v-infinity vector $\mathbf{v}_\infty^+$ and the two legs of the transfer orbit are as follows:

$$\mathbf{v}_\infty^- = \mathbf{v}_{fb} - \mathbf{v}_{to_1}$$

$$\mathbf{v}_\infty^+ = \mathbf{v}_{to_2} - \mathbf{v}_{fb}$$

where

$\mathbf{v}_{fb}$ = heliocentric velocity vector of the flyby planet at the flyby date
$\mathbf{v}_{to_1}$ = heliocentric velocity vector of the first transfer orbit at the flyby date
$\mathbf{v}_{to_2}$ = heliocentric velocity vector of the second transfer orbit at the flyby date

The turn angle of the planet-centered trajectory during the flyby is determined from

$$\phi = 2\sin^{-1}\left(\frac{1}{1 + r_p\, v_\infty^2 / \mu}\right)$$

where r_p is the periapsis radius of the flyby hyperbola, v_∞ is the magnitude of the incoming v-infinity vector and μ is the gravitational constant of the flyby planet.

The maximum turn angle possible during a gravity assist flyby occurs when the spacecraft just gazes the planet's surface. It is given by

$$\phi_{\max} = 2\sin^{-1}\left(\frac{1}{1 + r_e\, v_\infty^2 / \mu}\right)$$

where r_e is the radius of the flyby planet. The semimajor axis and orbital eccentricity of the flyby hyperbola are given by

$$a = -\mu / \left|\mathbf{v}_\infty^-\right|^2 = -\mu / \left|\mathbf{v}_\infty^+\right|^2$$

$$e = -1/\cos\theta_\infty = 1 - \frac{r_p}{a} = 1 + \frac{r_p\, v_\infty^2}{\mu}$$

where θ_∞ is the true anomaly at infinity which is determined from the following expression:

$$\theta_\infty = \frac{\pi}{2} - \frac{1}{2}\sin^{-1}\left(\frac{\left|\mathbf{v}_\infty^- \times \mathbf{v}_\infty^+\right|}{\left|\mathbf{v}_\infty^-\right|\left|\mathbf{v}_\infty^+\right|}\right)$$

The periapsis radius of the flyby hyperbola is determined from the expression $r = a\left(1 - e\right)$ and the flyby altitude is $h = r - r_p$.

The heliocentric speed gained during the flyby and the heliocentric delta-v vector caused by the close encounter can be determined from the following two equations:

$$\Delta v = 2 v_\infty / e$$

$$\Delta \mathbf{v} = \mathbf{v}_h^- - \mathbf{v}_h^+$$

In the second equation $\mathbf{v}_h^-$ is the heliocentric velocity vector of the spacecraft prior to the flyby and $\mathbf{v}_h^+$ is the heliocentric velocity vector after the flyby.

This section describes the solution steps and numerical method used in this computer program. The software solves the following system of two nonlinear equations:

$$\left|\mathbf{v}_\infty^-\right| - \left|\mathbf{v}_\infty^+\right| = 0$$

$$h_{fb} - h_t = 0$$

The first equation is the v-infinity matching constraint and the second equation is the (positive) flyby altitude constraint. In the second expression h_{fb} is the actual flyby altitude and h_t is the user-defined or “targeted” flyby altitude.

Solution Steps

The control variables for this problem are the flyby and arrival calendar dates. The computational steps in the solution of the system of nonlinear equations for any combination of departure, flyby and arrival calendar dates are as follows:

(1) compute the state vector of the departure planet at the departure date

(2) compute the state vector of the flyby planet at the flyby date

(3) compute the state vector of the arrival planet at the arrival date

(4) solve Lambert's problem for the departure-to-flyby leg and determine the initial velocity vector of the first leg

(5) compute the departure launch energy

(6) propagate the first leg of the heliocentric transfer orbit to the flyby calendar date

(7) calculate the incoming v-infinity speed relative to the flyby planet

(8) solve Lambert's problem for the second heliocentric leg and determine the initial velocity vector of the second leg

(9) calculate the outgoing v-infinity speed relative to the flyby planet

(10) determine the actual flyby angle

(11) calculate the planet-centered hyperbolic orbital elements

(12) compute the flyby altitude (kilometers)

(13) form the system of nonlinear equations

The planetary ephemerides used in this computer program are mean orbital elements relative to the ecliptic and equinox of J2000. They are based on the algorithm described in chapter 30 of *Astronomical Algorithms* by Jean Meeus[4]. Each orbital element is represented by a cubic polynomial of the form

$$a_0 + a_1 T + a_2 T^2 + a_3 T^3$$

where the fundamental time argument *T* is given by

$$T = \frac{JED - 2451545}{36525}$$

In this expression *JED* is the Julian ephemeris date.

Lambert's Problem

Each leg of the interplanetary transfer orbit requires the solution of Lambert's *two-point boundary-value problem (TPBVP)*. In this program Lambert's problem is solved using the method due to Geza Gedeon[5]. This algorithm involves the iterative solution of the equation given by:

$$N(z) = \frac{1}{z|z|^{1/2}\,2^{1/2}} \left\{ \begin{array}{l} \dfrac{1-k}{2} m\pi + k\left[|z|^{1/2} - |z|^{1/2}(1-z)^{1/2} \right] \\ -\left[w|z|^{1/2} - w|z|^{1/2} - w|z|^{1/2}\left(1-w^2 z\right)^{1/2} \right] \end{array} \right\}$$

This equation is solved using a Newton-Raphson numerical method that requires the first derivative of $N(z)$ which is given by the following expression:

$$N'(z) = \frac{dN}{dz} = \frac{1}{|z|2^{1/2}} \left\{ \frac{k}{(1-z)^{1/2}} - \frac{w^3}{\left(1-w^2 z\right)^{1/2}} - \frac{3N(z)}{2^{1/2}} \right\}$$

In these two equations $z = s/2a$, *a* is the semimajor axis of the heliocentric transfer orbit, $s = (r_1 + r_2 + c)/2$, r_1 and r_2 are the heliocentric distances of the initial and final trajectory segment and *c* is the chord between these two locations.

The software examines the launch energy of each valid patched-conic solution during the grid search and saves the smallest value as the global optimum solution. After the search is complete and a valid solution is found, the software provides a complete mission summary and heliocentric trajectory graphics.

References and Bibliography

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6. J. E. Dennis, Jr. and R. S. Schnabel, *Numerical Methods for Unconstrained Optimization and Nonlinear Equations*, Prentice-Hall, Inc., 1983.

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