# A MATLAB Script for Interplanetary Gravity-Assist Trajectory Design and Optimization

This document describes a MATLAB script called `flyby_snopt` that can be used to design and optimize interplanetary trajectories that include a single gravity assist maneuver. The user specifies the launch, flyby and destination planets, and the desired flyby altitude. The algorithm also requires initial guesses for the launch, flyby and arrival calendar dates. This script searches for a *patched-conic* gravity-assist trajectory that satisfies the flyby mission constraints ($V_\infty$ matching and user-defined flyby altitude) and minimizes the launch, arrival or total *impulsive* delta-v for the mission. The type of delta-v optimization is specified by the user.

The planet positions and velocities are modeled using the JPL DE 421 ephemeris. The trajectory optimization is performed with the SNOPT 6.0 nonlinear programming (NLP) algorithm.

The `flyby_snopt` software also provides a graphic display of the planet orbits and the interplanetary transfer trajectory.

## Typical user interaction

The following is typical user interaction with this MATLAB application. This example is an Earth-Venus gravity-assist-Mars mission with a flyby altitude at Venus of 500 kilometers. The user inputs for this example are in bold font.

The software allows the user to specify an initial guess for the launch, flyby and arrival calendar dates, and lower and upper bounds on the actual launch, flyby and arrival calendar dates found during the optimization process. For any guess for launch time $t_L$ and user-defined launch time lower and upper bounds $\Delta t_l$ and $\Delta t_u$, the launch time $t$ is constrained as follows:

$$t_L - \Delta t_l \le t \le t_L + \Delta t_u$$

Likewise, for any guess for arrival time $t_A$ and user-defined arrival time bounds $\Delta t_l$ and $\Delta t_u$, the arrival time $t$ is constrained as follows:

$$t_A - \Delta t_l \le t \le t_A + \Delta t_u$$

For fixed launch, flyby or arrival times, the lower and upper bounds should be set to 1.0d-4.

```
program flyby_snopt

gravity-assist trajectory analysis


departure - calendar date guess

please input the calendar date
(1 <= month <= 12, 1 <= day <= 31, year = all digits!)
? 2,1,2009
```

```
please input the departure date search boundary in days
? 30


flyby - calendar date guess

please input the calendar date
(1 <= month <= 12, 1 <= day <= 31, year = all digits!)
? 6,1,2009

please input the flyby date search boundary in days
? 30


arrival - calendar date guess

please input the calendar date
(1 <= month <= 12, 1 <= day <= 31, year = all digits!)
? 2,1,2010

please input the arrival date search boundary in days
? 30

planet menu

  <1> Mercury
  <2> Venus
  <3> Earth
  <4> Mars
  <5> Jupiter
  <6> Saturn
  <7> Uranus
  <8> Neptune
  <9> Pluto

please select the departure planet
? 3

planet menu

  <1> Mercury
  <2> Venus
  <3> Earth
  <4> Mars
  <5> Jupiter
  <6> Saturn
  <7> Uranus
  <8> Neptune
  <9> Pluto

please select the flyby planet
? 2

planet menu

  <1> Mercury
```

```
  <2> Venus
  <3> Earth
  <4> Mars
  <5> Jupiter
  <6> Saturn
  <7> Uranus
  <8> Neptune
  <9> Pluto

please select the arrival planet
? 4

please input the flyby altitude (kilometers)
? 500

      optimization menu

<1> minimize departure delta-v

<2> minimize arrival delta-v

<3> minimize total delta-v

selection (1, 2 or 3)
? 3
```

## Optimal solution and trajectory graphics

This section summarizes the program output for this example. The software provides the heliocentric orbital elements of each leg of the transfer trajectory in the Earth mean ecliptic and equinox of J2000 coordinate system. These numerical results also include the characteristics of the hyperbolic flyby trajectory with respect to the flyby planet as well as the trajectory characteristics at the times of entrance to the sphere-of-influence (SOI) and closest approach relative to the flyby planet. The time scale is Barycentric Dynamical Time (TDB).

```
program flyby_snopt

gravity assist trajectory analysis


LAUNCH CONDITIONS
=================

departure TDB calendar date    26-Jan-2009

departure TDB time             22:46:59.800

departure TDB Julian Date    2454858.449303

heliocentric launch delta-v vector and magnitude
(Earth mean ecliptic and equinox of J2000)
------------------------------------------

launch delta-vx             3598.330984 meters/second
launch delta-vy             -107.513676 meters/second
launch delta-vz            -4086.546917 meters/second

launch delta-v              5446.045425 meters/second

launch hyperbola characteristics
(Earth mean equator and equinox of J2000)
```

```
-----------------------------------------

asymptote right ascension     22.993112 degrees

asymptote declination        -44.131177 degrees

launch energy                 29.659411 (km/sec)^2

post-impulse heliocentric orbital elements and state vector of the spacecraft
(Earth mean ecliptic and equinox of J2000)
------------------------------------------

      sma (km)        eccentricity     inclination (deg)    argper (deg)
 1.2915349679e+008  1.5837709388e-001  8.4699120851e+000  3.3651002963e+002

     raan (deg)     true anomaly (deg)   arglat (deg)       period (days)
 3.0699636094e+002  2.0349519046e+002  1.8000522009e+002  2.9300131023e+002

        rx (km)                ry (km)                rz (km)                rmag (km)
 -8.86566261862594e+007  +1.17644805517813e+008  -1.97679458855093e+003  +1.47310208865380e+008

        vx (kps)               vy (kps)               vz (kps)               vmag (kps)
 -2.06860993612314e+001  -1.81557972184610e+001  -4.08724078800003e+000  +2.78254059571411e+001

heliocentric orbital elements and state vector of the departure planet
(Earth mean ecliptic and equinox of J2000)
------------------------------------------

      sma (km)        eccentricity     inclination (deg)    argper (deg)
 1.4973168636e+008  1.7489829256e-002  1.5179786566e-003  1.8766357058e+002

     raan (deg)     true anomaly (deg)   arglat (deg)       period (days)
 2.7657006597e+002  2.2767887557e+001  2.1043145813e+002  3.6574709272e+002

        rx (km)                ry (km)                rz (km)                rmag (km)
 -8.86566261862594e+007  +1.17644805517813e+008  -1.97679458855093e+003  +1.47310208865380e+008

        vx (kps)               vy (kps)               vz (kps)               vmag (kps)
 -2.42844303454548e+001  -1.80482835421561e+001  -6.93870616159664e-004  +3.02568024831238e+001


FLYBY CONDITIONS
================

flyby TDB calendar date        02-Jun-2009

flyby TDB time                 08:17:49.200

flyby TDB Julian Date        2454984.845708

launch-to-flyby time         126.396405 days

v-infinity in               7461.895071 meters/second
v-infinity out              7461.895266 meters/second

flyby altitude               499.999868 kilometers

maximum turn angle            58.793028 degrees
actual turn angle             56.203197 degrees

heliocentric delta-v        7029.649832 meters/second

max heliocentric delta-v    7326.580266 meters/second

planet-centered orbital elements and state vector of the spacecraft at periapsis
(Earth mean ecliptic and equinox of J2000)
------------------------------------------

      sma (km)        eccentricity     inclination (deg)    argper (deg)
 -5.8343985365e+003  2.1229777717e+000  7.2580123818e+001  3.8961669680e+001
```

```
     raan (deg)     true anomaly (deg)   arglat (deg)
 2.3728101950e+002  0.0000000000e+000  3.8961669680e+001

        rx (km)                 ry (km)                 rz (km)                 rmag (km)
 -1.71602837531439e+003  -4.95285439259147e+003  +3.93088690478186e+003  +6.55189986776983e+003

        vx (kps)                vy (kps)                vz (kps)                vmag (kps)
 +6.66637906058628e+000  +5.01736718015269e+000  +9.23200936203104e+000  +1.24436562175482e+001


b-plane coordinates of incoming hyperbola
(Earth mean ecliptic and equinox of J2000)
-----------------------------------------

b-magnitude                10926.123827  kilometers
b dot r                    -5624.821633
b dot t                     9367.046678
b-plane angle                329.015588  degrees
v-infinity                  7461.895071  meters/second
r-periapsis                 6551.899868  kilometers
decl-asymptote                69.561604  degrees
rasc-asymptote               359.934065  degrees

heliocentric orbital elements and state vector of the flyby planet
(Earth mean ecliptic and equinox of J2000)
-----------------------------------------

      sma (km)        eccentricity     inclination (deg)    argper (deg)
 1.0820914211e+008  6.7937075119e-003  3.3945276791e+000  5.4791401614e+001

     raan (deg)     true anomaly (deg)   arglat (deg)       period (days)
 7.6652371284e+001  1.6186109023e+002  2.1665249184e+002  2.2470163874e+002

        rx (km)                 ry (km)                 rz (km)                 rmag (km)
 +4.29752824588006e+007  -9.99954916183663e+007  -3.84950418649687e+006  +1.08907262975865e+008

        vx (kps)                vy (kps)                vz (kps)                vmag (kps)
 +3.19380484864012e+001  +1.37091444690867e+001  -1.65551062176933e+000  +3.47954062860716e+001


ARRIVAL CONDITIONS
==================

arrival TDB calendar date      14-Jan-2010

arrival TDB time               03:40:25.797

arrival TDB Julian Date      2455210.653076

flyby-to-arrival time        225.807368 days

heliocentric arrival delta-v vector and magnitude
(Earth mean ecliptic and equinox of J2000)
-----------------------------------------

arrival delta-vx             3467.201759 meters/second
arrival delta-vy             2289.215414 meters/second
arrival delta-vz             -755.119378 meters/second

arrival delta-v              4222.819026 meters/second

pre-impulse heliocentric orbital elements and state vector of the spacecraft
(Earth mean ecliptic and equinox of J2000)
-----------------------------------------

      sma (km)        eccentricity     inclination (deg)    argper (deg)
 1.7688276799e+008  3.9074776462e-001  2.4686070296e+000  3.2054723491e+002

     raan (deg)     true anomaly (deg)   arglat (deg)       period (days)
 3.4838070964e+002  1.7366712574e+002  1.3421436065e+002  4.6961136376e+002
```

```
       rx (km)                 ry (km)                 rz (km)                 rmag (km)
 -1.32035624490053e+008  +2.06286504624848e+008  +7.56469827644770e+006  +2.45040308489908e+008

       vx (kps)                vy (kps)                vz (kps)                vmag (kps)
 -1.60226322091160e+001  -8.71332573270333e+000  -5.07076868727946e-001  +1.82456546932027e+001
```

post-impulse heliocentric orbital elements and state vector of the spacecraft
(Earth mean ecliptic and equinox of J2000)
------------------------------------------

```
      sma (km)        eccentricity     inclination (deg)    argper (deg)
 2.2794303068e+008  9.3339698744e-002  1.8489037794e+000  2.8655658337e+002

     raan (deg)     true anomaly (deg)   arglat (deg)       period (days)
 4.9525421522e+001  1.4654795299e+002  7.3104536361e+001  6.8698900475e+002

       rx (km)                 ry (km)                 rz (km)                 rmag (km)
 -1.32035624490053e+008  +2.06286504624848e+008  +7.56469827644770e+006  +2.45040308489908e+008

       vx (kps)                vy (kps)                vz (kps)                vmag (kps)
 -1.94898339683834e+001  -1.10025411468527e+001  +2.48042509322756e-001  +2.23823829135722e+001
```

heliocentric orbital elements and state vector of the arrival planet
(Earth mean ecliptic and equinox of J2000)
------------------------------------------

```
      sma (km)        eccentricity     inclination (deg)    argper (deg)
 2.2794303068e+008  9.3339698744e-002  1.8489037794e+000  2.8655658337e+002

     raan (deg)     true anomaly (deg)   arglat (deg)       period (days)
 4.9525421522e+001  1.4654795299e+002  7.3104536361e+001  6.8698900475e+002

       rx (km)                 ry (km)                 rz (km)                 rmag (km)
 -1.32035624490001e+008  +2.06286504624876e+008  +7.56469827644935e+006  +2.45040308489904e+008

       vx (kps)                vy (kps)                vz (kps)                vmag (kps)
 -1.94898339683872e+001  -1.10025411468467e+001  +2.48042509322977e-001  +2.23823829135726e+001
```

MISSION SUMMARY
===============

```
total delta-v               9668.864451 meters/second

total energy                  47.491611 (km/sec)^2

total mission duration       352.203773 days
```

SOI ENTRANCE CONDITIONS
=======================

```
TDB calendar date      01-Jun-2009

TDB time               09:20:50.417

TDB Julian Date      2454983.889472
```

planet-centered orbital elements and state vector of the spacecraft
(Earth mean ecliptic and equinox of J2000)
------------------------------------------

```
      sma (km)        eccentricity     inclination (deg)    argper (deg)
 -5.9511194446e+003  1.0019810339e+000  1.0559757015e+002  2.5301764987e+002

     raan (deg)     true anomaly (deg)   arglat (deg)
 4.8426708131e+001  1.8363818418e+002  7.6655834048e+001

       rx (km)                 ry (km)                 rz (km)                 rmag (km)
 +2.15002548438914e+005  -5.80406211733818e+002  +5.77556116222914e+005  +6.16277129295818e+005
```

```
        vx (kps)               vy (kps)              vz (kps)               vmag (kps)
 -2.60506298761315e+000  +3.56958434068488e-003  -6.98968590934492e+000  +7.45936156937437e+000


CLOSEST APPROACH CONDITIONS
===========================

TDB calendar date      02-Jun-2009

TDB time               07:32:25.677

TDB Julian Date      2454984.814186

planet-centered orbital elements and state vector of the spacecraft
(Earth mean ecliptic and equinox of J2000)
------------------------------------------

      sma (km)        eccentricity     inclination (deg)    argper (deg)
 -5.9498809746e+003  1.0022331397e+000  1.0680922944e+002  2.5437511796e+002

     raan (deg)     true anomaly (deg)   arglat (deg)
 5.4097013129e+001  8.8716763841e-007  2.5437511885e+002

        rx (km)                ry (km)               rz (km)                rmag (km)
 -5.09594809961748e+000  -7.28797097893550e-001  -1.22491751575514e+001  +1.32869155293570e+001

        vx (kps)               vy (kps)              vz (kps)               vmag (kps)
 +1.10993117621459e+002  +1.82701652104185e+002  -5.70460961203619e+001  +2.21254656275304e+002


b-plane coordinates at closest approach
(Earth mean ecliptic and equinox of J2000)
------------------------------------------

b-magnitude                  397.853835  kilometers
b dot r                     -223.016638
b dot t                     -329.471172
b-plane angle                214.093785  degrees
v-infinity                  7389.125529  meters/second
r-periapsis                   13.286916  kilometers
decl-asymptote               -69.561158  degrees
rasc-asymptote               179.940197  degrees

flight path angle              0.000000  degrees
```

After the solution is displayed, the software will ask the user if he or she would like to create a graphics display of the transfer trajectory with the following prompt:

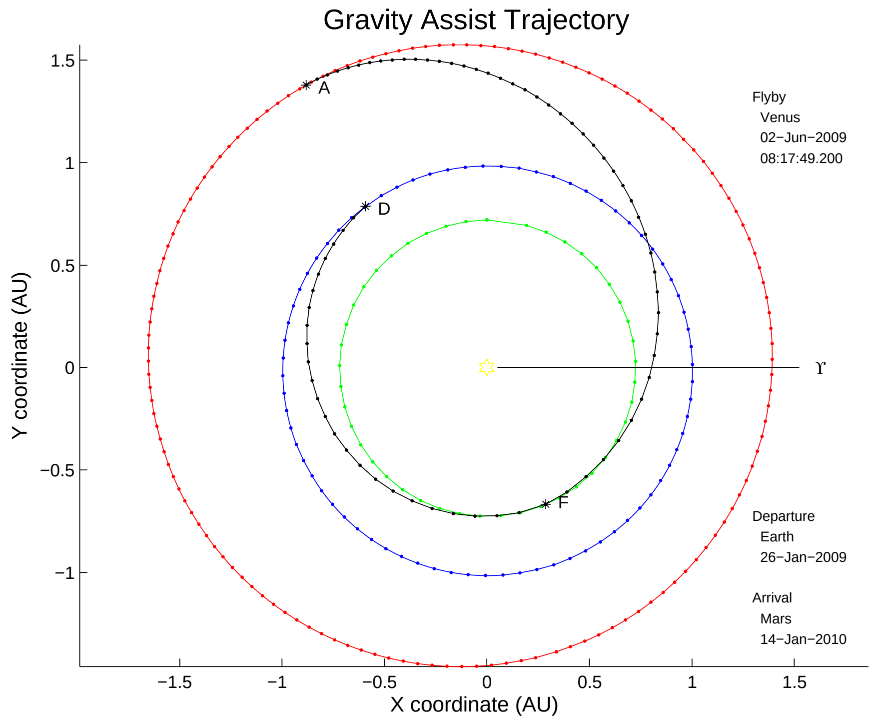
```
would you like to plot this trajectory (y = yes, n = no)
?
```

If the user's response is y for yes, the script will request a plot step size with

```
please input the plot step size (days)
?
```

A plot step size between 5 and 10 days is recommended.

The following is the graphics display for this example.  This plot is a *north ecliptic* view where we are looking down on the ecliptic plane from the north celestial pole.  The vernal equinox direction is the labeled line pointing to the right, the launch planet is labeled with an L, the flyby planet is labeled with an F and the arrival planet is labeled with an A.  The location of the launch, flyby and arrival planets at the launch time is marked with an asterisk.  Please note that the axes are labeled in Astronomical Units.




## Technical discussion

The computational steps used to create an initial guess for the spacecraft state, and the launch and arrival delta-v characteristics are as follows:

(1) compute the state vector of the departure planet at the departure date initial guess

(2) compute the state vector of the flyby planet at the flyby date initial guess

(3) compute the state vector of the arrival planet at the arrival date initial guess

(4) solve Lambert's problem for the departure-to-flyby leg and determine the initial velocity vector of this leg

(5) compute the launch delta-v vector from the launch planet’s velocity vector and the initial velocity vector of the first leg of the transfer trajectory

(6) solve Lambert's problem for the second heliocentric leg and determine the initial and final velocity vectors of the second leg

(7) compute the arrival delta-v vector from the arrival planet’s velocity vector and the final velocity vector of the second leg of the transfer trajectory

(8) estimate the flyby incoming v-infinity vector from the flyby planet’s velocity vector and the final velocity vector of the first leg of the transfer trajectory

*Lambert's Problem*

Lambert's problem is concerned with the determination of an orbit that passes between two positions within a specified time-of-flight. This classic astrodynamic problem is also known as the orbital two-point boundary value problem (TPBVP).

The time to traverse a trajectory depends only upon the length of the semimajor axis *a* of the transfer trajectory, the sum $r_i + r_f$ of the distances of the initial and final positions relative to a central body, and the length *c* of the chord joining these two positions. This relationship can be stated as follows:

$$tof = tof\left(r_i + r_f, c, a\right)$$

From the following form of Kepler's equation

$$t - t_0 = \sqrt{\frac{a^3}{\mu}\left(E - e\sin E\right)}$$

we can write

$$t = \sqrt{\frac{a^3}{\mu}}\left[E - E_0 - e\left(\sin E - \sin E_0\right)\right]$$

where *E* is the eccentric anomaly associated with radius *r*, $E_0$ is the eccentric anomaly at $r_0$, and $t = 0$ when $r = r_0$.

At this point we need to introduce the following trigonometric sun and difference identities:

$$\sin\alpha - \sin\beta = 2\sin\frac{\alpha - \beta}{2}\cos\frac{a + \beta}{2}$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha - \beta}{2}\sin\frac{a + \beta}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha - \beta}{2}\cos\frac{a + \beta}{2}$$

If we let $E = \alpha$ and $E_0 = \beta$ and substitute the first trig identity into the second equation above, we have the following equation:

$$t = \sqrt{\frac{a^3}{\mu}}\left\{E - E_0 - 2\sin\frac{E - E_0}{2}\left(e\cos\frac{E + E_0}{2}\right)\right\}$$

With the two substitutions given by

$$e\cos\frac{E+E_0}{2}=\cos\frac{\alpha+\beta}{2}$$

$$\sin\frac{E-E_0}{2}=\sin\frac{\alpha-\beta}{2}$$

the time equation becomes

$$t=\sqrt{\frac{a^3}{\mu}\left\{(\alpha-\beta)-2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}\right\}}$$

From the elliptic relationships given by

$$r=a(1-e\cos E)$$

$$x=a(\cos E-e)$$

$$y=a\sin E\sqrt{1-e^2}$$

and some more manipulation, we have the following equations:

$$\cos\alpha=\left(1-\frac{r+r_0}{2a}\right)-\frac{c}{2a}=1-\frac{r+r_0+c}{2a}=1-\frac{s}{a}$$

$$\sin\beta=\left(1-\frac{r+r_0}{2a}\right)+\frac{c}{2a}=1-\frac{r+r_0-c}{2a}=1-\frac{s-c}{a}$$

This part of the derivation makes use of the following three relationships:

$$\cos\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}=1-\frac{r+r_0}{2}$$

$$\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2}=\sin\frac{E-E_0}{2}\sqrt{1-\left(e\cos\frac{E+E_0}{2}\right)^2}$$

$$\left(\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2}\right)^2=\left(\frac{x-x_0}{2a}\right)^2+\left(\frac{y-y_0}{2a}\right)^2=\left(\frac{c}{2a}\right)^2$$

With the use of the half angle formulas given by

$$\sin\frac{\alpha}{2}=\sqrt{\frac{s}{2a}}\qquad\qquad \sin\frac{\beta}{2}=\sqrt{\frac{s-c}{2a}}$$

and several additional substitutions, we have the time-of-flight form of Lambert’s theorem

$$t = \sqrt{\frac{a^3}{\mu}}\left[(\alpha - \beta) - (\sin\alpha - \sin\beta)\right]$$

A discussion about the angles $\alpha$ and $\beta$ can be found in "Geometrical Interpretation of the Angles $\alpha$ and $\beta$ in Lambert's Problem" by J. E. Prussing, AIAA *Journal of Guidance and Control*, Volume 2, Number 5, Sept.-Oct. 1979, pages 442-443.

The algorithm used in this MATLAB script is based on the method described in "A Procedure for the Solution of Lambert's Orbital Boundary-Value Problem" by R. H. Gooding, *Celestial Mechanics and Dynamical Astronomy* **48:** 145-165, 1990. This iterative solution is valid for elliptic, parabolic and hyperbolic transfer orbits which may be either posigrade or retrograde, and involve one or more revolutions about the central body.

*Gravity-assist flight mechanics*

The vector relationships between the incoming v-infinity vector $\mathbf{v}_\infty^-$, the outgoing v-infinity vector $\mathbf{v}_\infty^+$ and the two legs of the heliocentric transfer orbit are as follows:

$$\mathbf{v}_\infty^- = \mathbf{v}_{fb} - \mathbf{v}_{to_1}$$

$$\mathbf{v}_\infty^+ = \mathbf{v}_{to_2} - \mathbf{v}_{fb}$$

where

$\mathbf{v}_{fb} =$ heliocentric velocity vector of the flyby planet at the flyby date
$\mathbf{v}_{to_1} =$ heliocentric velocity vector of the first transfer orbit at the flyby date
$\mathbf{v}_{to_2} =$ heliocentric velocity vector of the second transfer orbit at the flyby date

The turn angle of the planet-centered trajectory during the flyby is determined from

$$\delta = \frac{\pi}{2} - \frac{1}{2}\sin^{-1}\left(\frac{\left|\mathbf{v}_\infty^- \times \mathbf{v}_\infty^+\right|}{\left|\mathbf{v}_\infty^-\right|\left|\mathbf{v}_\infty^+\right|}\right) = 2\sin^{-1}\left(\frac{1}{1 + r_p v_\infty^2 / \mu}\right)$$

where $r_p$ is the periapsis radius of the flyby hyperbola, $v_\infty$ is the magnitude of the incoming (or outgoing) v-infinity vector and $\mu$ is the gravitational constant of the flyby planet.

The maximum turn angle possible during a gravity assist flyby occurs when the spacecraft just gazes the planet's surface. It is given by

$$\delta_{\max} = 2\sin^{-1}\left(\frac{1}{1 + r_s v_\infty^2 / \mu}\right)$$

where $r_s$ is the radius of the flyby planet. The semimajor axis and orbital eccentricity of the flyby hyperbola are given by

$$a = -\mu / \left|\mathbf{v}_\infty^-\right|^2 = -\mu / \left|\mathbf{v}_\infty^+\right|^2$$

$$e = -1/\cos\theta_\infty = 1 - \frac{r_p}{a} = 1 + \frac{r_p\, v_\infty^2}{\mu}$$

where $\theta_\infty$ is the true anomaly at infinity which is determined from the following expression:

$$\theta_\infty = \frac{\pi}{2} - \frac{1}{2}\sin^{-1}\left(\frac{\left|\mathbf{v}_\infty^- \times \mathbf{v}_\infty^+\right|}{\left|\mathbf{v}_\infty^-\right|\left|\mathbf{v}_\infty^+\right|}\right)$$

The periapsis radius of the flyby hyperbola is determined from the expression $r = a\left(1 - e\right)$ and the flyby altitude is $h = r - r_p$.

The heliocentric speed gained during the flyby and the heliocentric delta-v vector caused by the close encounter can be determined from the following two equations:

$$\Delta v_{fb} = 2 v_\infty / e$$

$$\Delta \mathbf{v}_{fb} = \mathbf{v}_h^- - \mathbf{v}_h^+$$

where *e* is the orbital eccentricity of the hyperbolic flyby trajectory.

In the second equation $\mathbf{v}_h^-$ is the heliocentric velocity vector of the spacecraft prior to the flyby and $\mathbf{v}_h^+$ is the heliocentric velocity vector after the flyby. For any planet it can be shown that the *maximum* heliocentric delta-v possible is given by the expression

$$\Delta v_{\max} = \sqrt{\frac{\mu}{r_s}}$$

This corresponds to a “grazing” flyby at the planet’s surface and is equal to the “local circular velocity” for the flyby planet at the surface.

During the optimization analysis, the software enforces the following two nonlinear equality *point constraints*:

$$\left|\mathbf{v}_\infty^-\right| - \left|\mathbf{v}_\infty^+\right| = 0$$

$$h_{fb} - h_t = 0$$

The first equation is the v-infinity matching constraint and the second equation is the (positive) flyby altitude constraint. In the second expression $h_{fb}$ is the actual flyby altitude and $h_t$ is the user-defined or "targeted" flyby altitude.

The orientation of the departure hyperbolas is specified in terms of the right ascension and declination of the asymptote of the launch hyperbola. These coordinates can be calculated using the components of the planet-centered departure unit delta-v vector.

The right ascension of the asymptote is determined from

$$\alpha = \tan^{-1}\left(\Delta v_y, \Delta v_z\right)$$

and the geocentric declination of the asymptote is given by

$$\delta = 90^0 - \cos^{-1}\left(\Delta v_z\right)$$

where $\Delta v_x, \Delta v_y, \Delta v_z$ are the components of the unit $\Delta v$ vector.

In typical "targeting specs", the right ascension is called RLA and the declination is called DLA. The corresponding launch energy is called C3. C3 is equal to twice the specific (per unit mass) orbital energy.

In this script the heliocentric planetary coordinates and therefore the $\Delta v$ vectors are computed in the J2000 ecliptic and equinox coordinate system. In order to determine the orientation of the departure hyperbola, the delta-v vector must be transformed to the Earth equatorial frame.

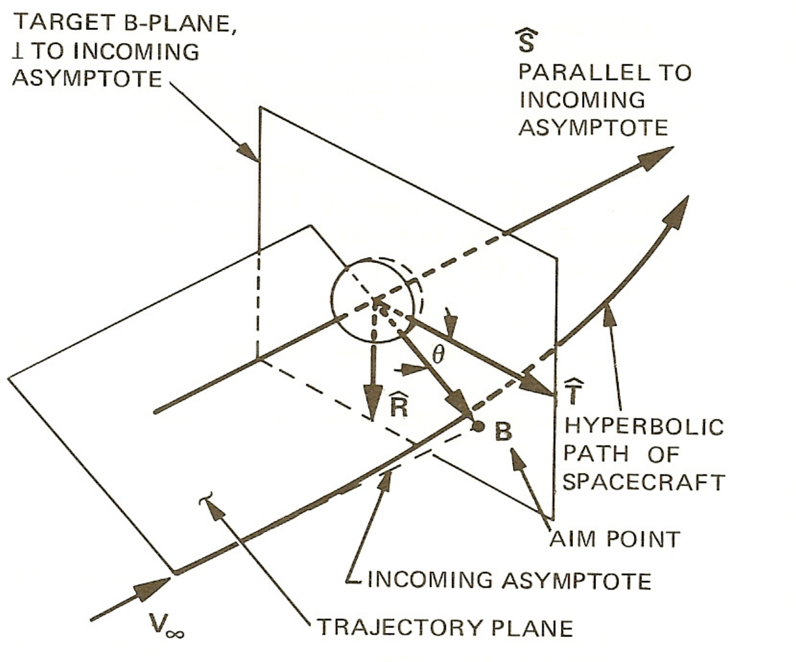
The required transformation is given by

$$\Delta \mathbf{V}_{eq} = \begin{bmatrix} 1 & \text{-0.000000479966} & 0 \\ 0.000000440360 & 0.917482137087 & 0.397776982902 \\ \text{-0.000000190919} & \text{-0.397776982902} & 0.917482137087 \end{bmatrix} \Delta \mathbf{V}_{ec}$$

where $\Delta \mathbf{V}_{ec}$ is the delta-velocity vector in the ecliptic frame, and $\Delta \mathbf{V}_{eq}$ is the delta-velocity vector in the Earth equatorial frame.

*The B-plane*

The derivation of B-plane coordinates is described in the classic JPL reports, "A Method of Describing Miss Distances for Lunar and Interplanetary Trajectories" and "Some Orbital Elements Useful in Space Trajectory Calculations", both by William Kizner. The following diagram illustrates the fundamental geometry of the B-plane coordinate system.




The arrival asymptote unit vector $\hat{\mathbf{S}}$ is given by

$$\hat{\mathbf{S}} = \begin{Bmatrix} \cos\delta_\infty \cos\alpha_\infty \\ \cos\delta_\infty \sin\alpha_\infty \\ \sin\delta_\infty \end{Bmatrix}$$

where $\delta_\infty$ and $\alpha_\infty$ are the declination and right ascension of the asymptote of the incoming or outgoing hyperbola.

The following computational steps summarize the calculation of the *predicted* B-plane vector from a planet-centered position vector **r** and velocity vector **v** at the sphere-of-influence.

angular momentum vector

$$\mathbf{h} = \mathbf{r} \times \mathbf{v}$$

radius rate

$$\dot{r} = \frac{\mathbf{r} \bullet \mathbf{v}}{|\mathbf{r}|}$$

semiparameter

$$p = \frac{h^2}{\mu}$$

semimajor axis

$$a = \frac{r}{\left(2 - \frac{r\,v^2}{\mu}\right)}$$

orbital eccentricity

$$e = \sqrt{1 - p/a}$$

true anomaly

$$\cos\theta = \frac{p - r}{e\,r}$$

$$\sin\theta = \frac{\dot{r}\,h}{e\mu}$$

B-plane magnitude

$$B = \sqrt{p|a|}$$

fundamental vectors

$$\hat{\mathbf{z}} = \frac{r\,\mathbf{v} - \dot{r}\,\mathbf{r}}{h}$$

$$\hat{\mathbf{p}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\mathbf{z}}$$

$$\hat{\mathbf{q}} = \sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\mathbf{z}}$$

S vector

$$\mathbf{S} = -\frac{a}{\sqrt{a^2 + b^2}}\,\hat{\mathbf{p}} + \frac{b}{\sqrt{a^2 + b^2}}\,\hat{\mathbf{q}}$$

B vector

$$\mathbf{B} = \frac{b^2}{\sqrt{a^2 + b^2}}\,\hat{\mathbf{p}} + \frac{ab}{\sqrt{a^2 + b^2}}\,\hat{\mathbf{q}}$$

T vector

$$\mathbf{T} = \frac{\left(S_y^2, -S_x^2, 0\right)^T}{\sqrt{S_x^2 + S_y^2}}$$

R vector

$$\mathbf{R} = \mathbf{S} \times \mathbf{T} = \left(-S_z T_y,\; S_z T_x,\; S_x T_y - S_y T_x\right)^T$$

*Calculating Sphere-of-Influence Entrance and Closest Approach*

This section describes the numerical methods used to predict the time at which the spacecraft enters the sphere-of-influence (SOI) of the flyby planet and the time of closest approach to the flyby planet. The search algorithm uses a numerical integration method embedded within a root-finding function. These calculations can be used to assess the difference between the patched-conic and numerically integrated solution for the flyby hyperbola.

The sphere-of-influence *objective function* is given by

$$\Delta = \left|\mathbf{r}_{p-sc}\right| - r_{SOI}$$

where $\mathbf{r}_{p-sc}$ is the radius vector of the spacecraft relative to the flyby planet and $r_{SOI}$ is the SOI radius of the flyby planet. The SOI radius is a function of the gravity of the flyby planet.

During the search for SOI conditions, the heliocentric equations of motion of the spacecraft subject to the point-mass gravity of the sun are defined by

$$\mathbf{a} = -\mu_s \frac{\mathbf{r}_{s-sc}}{\left|\mathbf{r}_{s-sc}\right|^3}$$

where $\mu_s$ is the gravitational constant of the sun and $\mathbf{r}_{s-sc}$ is the heliocentric position vector from the sun to the spacecraft.

The following is the MATLAB source code from the main script that predicts SOI entrance. It uses the root-finder option of the built-in `ODE45` function to evaluate the heliocentric equations of motion defined in a MATLAB function named `heqm` while searching for the time at which the flyby planet-to-spacecraft distance equals the SOI radius of the flyby planet.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% integrate trajectory from first impulse
% to SOI entrance at flyby planet
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% set up for ode45

options = odeset('RelTol', 1.0e-6, 'AbsTol', 1.0e-8, 'Events', @soi_event);

% define maximum search time (seconds)

tof = 86400.0 * (jdate2 - jdate1);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% solve for SOI entrance conditions
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

[t, ysol, tevent, yevent, ie] = ode45(@heqm, [0 tof], [rito1 vito1], options);
```

The following is the MATLAB source code that computes the scalar value of the SOI objective function.

```
function [value, isterminal, direction] = soi_event(t, y)

% entrance to SOI of flyby planet event function

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%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

global ip2 rsoi jdate1 rp2sc vp2sc

% current TDB julian date

jdate = jdate1 + t/ 86400.0d0;

% compute the position and velocity of flyby planet

[rp, vp] = p2000(ip2, jdate);

rsc = y(1:3);

vsc = y(4:6);

% vector from flyby planet to spacecraft

rp2sc = rp - rsc;

vp2sc = vp - vsc;

% objective function

value = norm(rp2sc) - rsoi(ip2);

isterminal = 1;

direction =  [];
```

The software uses a similar technique to predict the spacecraft flight conditions at closest approach to the flyby planet. The flyby planet-centered initial conditions for the search are determined from the state vector of the spacecraft at SOI entrance.

For close approach prediction, the script uses the following objective function

$$\Delta = \frac{\mathbf{r}_{p-sc} \bullet \mathbf{v}_{p-sc}}{\left| \mathbf{r}_{p-sc} \bullet \mathbf{v}_{p-sc} \right|}$$

which is simply the sine of the flight path angle. Since the spacecraft's orbit within the SOI is hyperbolic, periapsis occurs whenever the flight path angle is zero.

During the search for closest approach conditions, the flyby planet-centered equations of motion subject to the point-mass gravity of the sun and the flyby planet are computed using

$$\mathbf{a} = \mathbf{a}_p + \mathbf{a}_s = -\mu_p \frac{\mathbf{r}_{p-sc}}{\left|\mathbf{r}_{p-sc}\right|^3} - \mu_s \left\{ \frac{\mathbf{r}_{s-sc}}{\left|\mathbf{r}_{s-sc}\right|^3} + \frac{\mathbf{r}_p}{\left|\mathbf{r}_p\right|^3} \right\}$$

In this equation, **a** is the total acceleration on the spacecraft and $\mathbf{r}_{s-sc}$ is the heliocentric position vector of the spacecraft given by

$$\mathbf{r}_{s-sc} = \mathbf{r}_{p-sc} + \mathbf{r}_p$$

where $\mathbf{r}_p$ is the heliocentric position vector of the flyby planet.

## SNOPT algorithm implementation

This section provides details about the part of the `flyby_snopt` MATLAB script that solves this nonlinear programming (NLP) problem using the SNOPT 6.0 algorithm. In this classic trajectory optimization problem, the launch, flyby and arrival calendar dates are the *control variables* and the user-specified scalar $\Delta V$ is the *objective function* or *performance index*.

MATLAB versions of SNOPT 6.0 for several computer platforms can be found at Professor Philip Gill's web site which is located at http://scicomp.ucsd.edu/~peg/. A PDF version of the SNOPT user's manual is also available at this website.

The SNOPT algorithm requires an initial guess for the control variables. For this problem they are given by

```
% control variables initial guess

xg(1) = jdatei1 - jdate0;

xg(2) = jdatei2 - jdate0;

xg(3) = jdatei3 - jdate0;

xg = xg';
```

where `jdatei1`, `jdatei2` and `jdatei3` are the initial user-provided launch, flyby and arrival calendar date guesses, and `jdate0` is a reference Julian date equal to `2451544.5` (January 1, 2000). This offset value is used to *scale* the control variables.

The algorithm also requires lower and upper bounds for the control variables. These are determined from the user-defined initial guesses and search boundaries as follows:

```
% bounds on control variables

xlwr(1) = xg(1) - ddays1;
xupr(1) = xg(1) + ddays1;

xlwr(2) = xg(2) - ddays2;
xupr(2) = xg(2) + ddays2;

xlwr(3) = xg(3) – ddays3;
xupr(3) = xg(3) + ddays3;
```

```
xlwr = xlwr';
xupr = xupr';

xlwr = xlwr';
xupr = xupr';
```

where `ddays1`, `ddays2` and `ddays3` are the user-defined launch, flyby and arrival search boundaries, respectively.

The algorithm also requires lower and upper bounds on the objective function. For this problem these bounds are given by

```
% bounds on objective function

flow(1) = 0.0d0;
fupp(1) = +Inf;
```

The following MATLAB source code enforces the equality flyby constraints:

```
% load mission constraints

y(2) = fbalt - alttar;

y(3) = vinfm_in - vinfm_out;
```

In this code `fbalt`, `vinfm_in`, and `vinfm_out` are the current flyby altitude, incoming and outgoing v-infinity speeds, respectively. `alttar` is the user-defined flyby altitude.

The actual call to the SNOPT MATLAB interface function is as follows

```
[x, f, inform, xmul, fmul] = snopt(xg, xlwr, xupr, flow, fupp, 'fbyfunc');
```

where `fbyfunc` is the name of the MATLAB function that solves Lambert's problem and computes the current value of the objective function.

The `flyby_snopt` script will also create summary files and set other algorithm options such as derivative and iteration limits. The MATLAB code that performs these functions is as follows:

```
% set SNOPT options

snprintfile('flyby_snopt.out');

snsummary('flyby_snopt.sum');

snseti('derivative option', 0);

snseti('minor iterations limit', 1000);
```

## References and bibliography

"Gravity-Assist Trajectory Design with MATLAB", C. David Eagle, AAS 99-138, AAS/AIAA Space Flight Mechanics Meeting, Breckenridge, CO, February 7-10, 1999.

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"Modern Astrodynamics", Victor R. Bond and Mark C. Allman, Princeton University Press, 1996.

"User's Guide for SNOPT Version 6, A Fortran Package for Large-Scale Nonlinear Programming", Philip E. Gill, Walter Murray and Michael A. Saunders, December 2002.

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"Gravitational Assist in Celestial Mechanics – A Tutorial", J. A. Van Allen, American Journal of Physics, **71** (5), May 2003.

"Notes for the Gravitational Assisted Trajectories Lectures", E. Barrabes, G. Gomez, and J. Rodriquez-Canabal, Advanced Topics in Astrodynamics Summer Course, Barcelona, July 2004.

# APPENDIX A

## Contents of the Simulation Summary

This appendix is a brief summary of the information contained in the simulation summary screen displays produced by the `flyby_snopt` software.

The simulation summary screen display contains the following information:

**calendar date** = calendar date of trajectory event

**TDB time** = TDB time of trajectory event

**TDB Julian Date** = julian date of trajectory event on TDB time scale

**sma (km)** = semimajor axis in kilometers

**eccentricity** = orbital eccentricity (non-dimensional)

**inclination (deg)** = orbital inclination in degrees

**argper (deg)** = argument of periapsis in degrees

**raan (deg)** = right ascension of the ascending node in degrees

**true anomaly (deg)** = true anomaly in degrees

**arglat (deg)** = argument of latitude in degrees. The argument of latitude is the sum of true anomaly and argument of perigee.

**period (days)** = orbital period in days

**rx (km)** = x-component of the spacecraft's position vector in kilometers

**ry (km)** = y-component of the spacecraft's position vector in kilometers

**rz (km)** = z-component of the spacecraft's position vector in kilometers

**rmag (km)** = scalar magnitude of the spacecraft's position vector in kilometers

**vx (kps)** = x-component of the spacecraft's velocity vector in kilometers per second

**vy (kps)** = y-component of the spacecraft's velocity vector in kilometers per second

**vz (ksp)** = z-component of the spacecraft's velocity vector in kilometers per second

**vmag (kps)** = scalar magnitude of the spacecraft's velocity vector in kilometers per second

**deltav-x** = x-component of the impulsive TCM velocity vector in meters/second

**deltav-y** = y-component of the impulsive TCM velocity vector in meters/second

**deltav-z** = z-component of the impulsive TCM velocity vector in meters/second

**delta-v** = scalar magnitude of the impulsive TCM delta-v in meters/seconds

**b-magnitude** = magnitude of the b-plane vector

**b dot r** = dot product of the b-vector and r-vector

**b dot t** = dot product of the b-vector and t-vector

**theta** = orientation of the b-plane vector in degrees

**v-infinity** = magnitude of incoming v-infinity vector in kilometers/second

**r-periapsis** = periapsis radius of incoming hyperbola in kilometers

**decl-asymptote** = declination of incoming v-infinity vector in degrees

**rasc-asymptote** = right ascension of incoming v-infinity vector in degrees

**fpa** = flight path angle in degrees