A Computer Program for Patched-Conic, Ballistic Interplanetary Trajectory Design and Optimization

This document is the user's manual for a Fortran computer program called `ipto_ftn` that solves the classic two-impulse patched-conic, ballistic interplanetary trajectory optimization problem between two celestial bodies in our solar system. The software attempts to minimize the scalar magnitude of the departure delta-v, the arrival delta-v or the total delta-v vectors for the interplanetary transfer. The type of trajectory optimization is specified by the user. The software will also perform a graphical primer vector analysis of the solution.

The important features of this scientific simulation are as follows:

- *patched-conic* interplanetary trajectory modeling
- impulsive maneuver modeling
- elliptical, non-coplanar asteroid and comet orbits
- JPL DE421 planetary ephemeris model

Program Execution

An input file created by the user can be run from the command line or a simple batch file with a statement similar to the following:

```
ipto_ftn mars09.in
```

If the software is executed without an input file on the command line, the computer program will display the following title screen and file name prompt:

```
**********************************
*        program ipto_ftn        *
*                                *
*    ballistic interplanetary    *
*     trajectory optimization    *
*                                *
*        December 5, 2011        *
**********************************

please input the name of the simulation definition file
```

The user should respond to this prompt with the name of a compatible input data file including the filename extension.

To create a DOS command window in Windows 7, select **start**, then **All Programs**, then **Accessories** and finally **Command Prompt**. The size, font and other characteristics of the screen can be controlled by the user with the **c:** icon in the upper left corner of the window. To log into the subdirectory created during the installation of the Fortran executable and support files, type **root:** and then **cd subdirectory** from the DOS command line where root is the name of the root directory, usually c:, and subdirectory is the name of the subdirectory created by the user. The DOS command line prompt looks similar to **C:\ipto_ftn>_.**

Input File Format and Contents

The `ipto_ftn` computer program is "data-driven" by a simple user-created text file. The following is a typical input or "simulation definition" file used by the software. This example is an Earth-to-Mars ballistic rendezvous trajectory that minimizes the total delta-v for the mission.

In the following discussion the actual input file contents are in `courier` font and all explanations are in *times italic* font. Please note that the fundamental time argument in this computer program is Barycentric Dynamical Time (TDB).

Each data item within an input file is preceded by one or more lines of *annotation* text. Do not delete any of these annotation lines or increase or decrease the number of lines reserved for each comment. However, you may change them to reflect your own explanation. The annotation line also includes the correct units and when appropriate, the valid range of the input. ASCII text input is <u>not</u> case sensitive but must be spelled correctly.

The software allows the user to specify an initial guess for the departure and arrival calendar dates and lower and upper bounds on the actual dates found during the optimization process. For any guess for departure time t_L and user-defined departure time lower and upper bounds Δt_l and Δt_u, the departure time t is constrained as follows:

$$t_L - \Delta t_l \le t \le t_L + \Delta t_u$$

Likewise, for any guess for arrival time t_A and user-defined arrival time bounds Δt_l and Δt_u, the arrival time t is constrained as follows:

$$t_A - \Delta t_l \le t \le t_A + \Delta t_u$$

The departure and arrival times are the control variables for this nonlinear programming (NLP) problem.

The first six lines of any input file are reserved for user comments. These lines are ignored by the software. However the input file must begin with six and only six initial text lines.

```
*****************************************
** patched-conic ballistic interplanetary
** trajectory design and optimization
** Mars '09 example - mars09.in
** July 12, 2011
*****************************************
```

The first program input is an integer that defines the type of delta-v optimization. Please note that option 4, `no optimization` *simply solves the two-body, orbital two-point boundary value problem.*

```
*******************
* simulation type *
*******************
 1 = minimize departure delta-v
 2 = minimize arrival delta-v
 3 = minimize total delta-v
 4 = no optimization
--------------------
3
```

The next three inputs are the user's initial guess for the departure calendar date. Be sure to include all four digits of the calendar year.

```
departure calendar date initial guess (month, day, year)
9, 24, 2009
```

The next two inputs are the lower and upper bounds for the departure calendar date search interval. These values should be input in days.

```
departure date search boundary (days)
-60, +60
```

The next program input is an integer that specifies the departure planet.

```
********************
* departure planet *
********************
 1 = Mercury
 2 = Venus
 3 = Earth
 4 = Mars
 5 = Jupiter
 6 = Saturn
 7 = Uranus
 8 = Neptune
 9 = Pluto
----------
3
```

The next set of inputs defines the user's initial guess for the arrival calendar date, the search interval and the arrival planet or comet/asteroid.

```
arrival calendar date initial guess (month, day, year)
7, 10, 2010

arrival date search boundary (days)
-60, +60

**************************
* arrival celestial body *
**************************
 1 = Mercury
 2 = Venus
 3 = Earth
 4 = Mars
 5 = Jupiter
 6 = Saturn
 7 = Uranus
 8 = Neptune
 9 = Pluto
 0 = asteroid/comet
-------------------
4
```

The next series of inputs include the name and classical orbital elements of a comet or asteroid (arrival celestial body = 0). Please note that the angular orbital elements must be specified with respect to a heliocentric, Earth mean ecliptic and equinox of J2000 coordinate system. The calendar date of perihelion passage should be with respect to the TDB time system.

```
*********************************************
* asteroid/comet classical orbital elements *
* (heliocentric, Earth mean ecliptic J2000) *
```

```
******************************************

asteroid/comet name
Tempel 1

calendar date of perihelion passage (month, day, year)
7, 5.3153, 2005

perihelion distance (au)
1.506167

orbital eccentricity (non-dimensional)
0.517491

orbital inclination (degrees)
10.5301

argument of perihelion (degrees)
178.8390

longitude of the ascending node (degrees)
68.9734
```

The final two inputs are the altitude and orbital inclination of the initial circular Earth park orbit.

```
****************************
geocentric trajectory modeling
****************************

circular park orbit altitude (kilometers)
185.32

circular park orbit inclination (degrees)
28.5
```

Program Solution

The following is the program output created by the `ipto_ftn` simulation for this example. Please note that the departure and arrival hyperbola coordinates are with respect to the Earth mean equator and equinox of J2000 (EME2000) system. Please see Appendix A of this document for additional details about this information.

```
program ipto_ftn - interplanetary mission design
================================================

input data file ==> mars09.in

minimize total delta-v


heliocentric coordinates of the planet at departure
(Earth mean ecliptic and equinox of J2000)
------------------------------------------

calendar date           October 14, 2009

TDB time                14:36:32.035

TDB Julian date         2455119.10870411

     sma (au)            eccentricity       inclination (deg)       argper (deg)
 0.100060820685D+01    0.164776843710D-01    0.808465706362D-03    0.374699482583D+02
```

raan (deg)	true anomaly (deg)	arglat (deg)	period (days)
0.633326682202D+02	0.280446308313D+03	0.317916256571D+03	0.365590176608D+03

rx (km)	ry (km)	rz (km)	rmag (km)
0.139058874109D+09	0.540740344397D+08	-.141100894780D+04	0.149202451960D+09

vx (kps)	vy (kps)	vz (kps)	vmag (kps)
-.112747728030D+02	0.276631299022D+02	0.317355663847D-03	0.298725502400D+02

spacecraft heliocentric coordinates after the first impulse
(Earth mean ecliptic and equinox of J2000)
----

calendar date October 14, 2009

TDB time 14:36:32.035

TDB Julian date 2455119.10870411

sma (au)	eccentricity	inclination (deg)	argper (deg)
0.129413047808D+01	0.229680280449D+00	0.135358573464D+00	0.184267871988D+03

raan (deg)	true anomaly (deg)	arglat (deg)	period (days)
0.201019566919D+03	0.355961486527D+03	0.180229358515D+03	0.537731558232D+03

rx (km)	ry (km)	rz (km)	rmag (km)
0.139058874109D+09	0.540740344397D+08	-.141100894780D+04	0.149202451960D+09

vx (kps)	vy (kps)	vz (kps)	vmag (kps)
-.123888187414D+02	0.306588953543D+02	-.781087306020D-01	0.330674582502D+02

spacecraft heliocentric coordinates prior to the second impulse
(Earth mean ecliptic and equinox of J2000)
----

calendar date September 3, 2010

TDB time 06:34:10.704

TDB Julian date 2455442.77373500

sma (au)	eccentricity	inclination (deg)	argper (deg)
0.129413047808D+01	0.229680280449D+00	0.135358573464D+00	0.184267871988D+03

raan (deg)	true anomaly (deg)	arglat (deg)	period (days)
0.201019566919D+03	0.202357224937D+03	0.266250969248D+02	0.537731558232D+03

rx (km)	ry (km)	rz (km)	rmag (km)
-.156874862616D+09	-.172068693183D+09	0.246522313449D+06	0.232846340894D+09

vx (kps)	vy (kps)	vz (kps)	vmag (kps)
0.172402027656D+02	-.125374179635D+02	0.422572366854D-01	0.213169703820D+02

spacecraft heliocentric coordinates after the second impulse
(Earth mean ecliptic and equinox of J2000)
----

calendar date September 3, 2010

TDB time 06:34:10.704

TDB Julian date 2455442.77373500

sma (au)	eccentricity	inclination (deg)	argper (deg)
0.152366649381D+01	0.933319170358D-01	0.184892807903D+01	0.286616590221D+03
raan (deg)	true anomaly (deg)	arglat (deg)	period (days)
0.495240871537D+02	0.251502944093D+03	0.178119534315D+03	0.686962930939D+03
rx (km)	ry (km)	rz (km)	rmag (km)
-.156874862616D+09	-.172068693183D+09	0.246522313449D+06	0.232846340894D+09
vx (kps)	vy (kps)	vz (kps)	vmag (kps)
0.188147005757D+02	-.142516833461D+02	-.760643083064D+00	0.236152919493D+02

heliocentric coordinates of celestial body at arrival
(Earth mean ecliptic and equinox of J2000)

----

calendar date September 3, 2010

TDB time 06:34:10.704

TDB Julian date 2455442.77373500

sma (au)	eccentricity	inclination (deg)	argper (deg)
0.152366649381D+01	0.933319170371D-01	0.184892807903D+01	0.286616590220D+03
raan (deg)	true anomaly (deg)	arglat (deg)	period (days)
0.495240871544D+02	0.251502944094D+03	0.178119534315D+03	0.686962930939D+03
rx (km)	ry (km)	rz (km)	rmag (km)
-.156874862613D+09	-.172068693184D+09	0.246522313454D+06	0.232846340893D+09
vx (kps)	vy (kps)	vz (kps)	vmag (kps)
0.188147005759D+02	-.142516833459D+02	-.760643083065D+00	0.236152919493D+02

IMPULSIVE MANEUVER SUMMARY

----

departure heliocentric delta-v vector and magnitude
(Earth mean ecliptic and equinox of J2000)

----

x-component of delta-v	-1114.04593837300	meters/second
y-component of delta-v	2995.76545217820	meters/second
z-component of delta-v	-78.4260862658114	meters/second
delta-v magnitude	3197.16431361869	meters/second

arrival heliocentric delta-v vector and magnitude
(Earth mean ecliptic and equinox of J2000)

----

x-component of delta-v	1574.49781006571	meters/second
y-component of delta-v	-1714.26538258882	meters/second
z-component of delta-v	-802.900319749633	meters/second
delta-v magnitude	2462.19375340329	meters/second

HYPERBOLIC TRAJECTORY CHARACTERISTICS
(Earth mean equator and equinox of J2000)

```
departure hyperbola
-------------------

c3                    10.2218596482768      km**2/sec**2

v-infinity            3197.16431361869      meters/second

decl-asymptote        20.5004107372075      degrees

rasc-asymptote        111.839450117695      degrees


arrival hyperbola
-----------------

c3                    6.06239807929820      km**2/sec**2

v-infinity            2462.19375340329      meters/second

decl-asymptote       -35.1787575879296      degrees

rasc-asymptote        321.477235067672      degrees


time of flight        323.665030893870      days
```

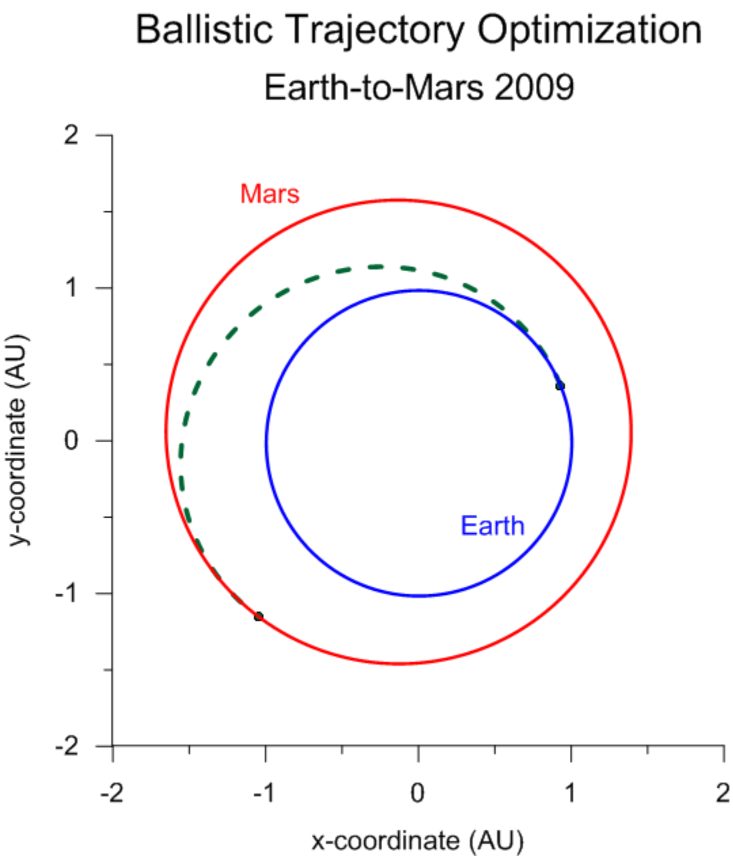
The final part of the screen display summarizes the characteristics of the geocentric park orbit and launch hyperbola.

```
orbital elements and state vector of park orbit at injection
(Earth mean equator and equinox of J2000)
-----------------------------------------

     sma (km)         eccentricity       inclination (deg)     argper (deg)
 0.6563460000D+04    0.0000000000D+00    0.2000000000D+02    0.0000000000D+00

    raan (deg)      true anomaly (deg)     arglat (deg)        period (min)
 0.2183944940D+02    0.3012258027D+03    0.3012258027D+03    0.8819805229D+02

     rx (km)             ry (km)             rz (km)            rmag (km)
 0.5120389548D+04    -.3629827221D+04    -.1919628192D+04    0.6563460000D+04

     vx (kps)            vy (kps)            vz (kps)           vmag (kps)
 0.4773465278D+01    0.6002918319D+01    0.1381749554D+01    0.7792960344D+01


orbital elements and state vector of hyperbola at injection
(Earth mean equator and equinox of J2000)
-----------------------------------------

     sma (km)         eccentricity       inclination (deg)     argper (deg)
 -.3899490457D+05    0.1168315719D+01    0.2051630458D+02    0.3033433722D+03

    raan (deg)      true anomaly (deg)     arglat (deg)        period (min)
 0.1948385721D+02    0.9237038576D-01    0.3034357426D+03    0.1666666667D+98

     rx (km)             ry (km)             rz (km)            rmag (km)
 0.5120389548D+04    -.3629827221D+04    -.1919628192D+04    0.6563460000D+04

     vx (kps)            vy (kps)            vz (kps)           vmag (kps)
 0.7060395523D+01    0.8771279403D+01    0.2213096068D+01    0.1147529178D+02


hyperbolic injection delta-v vector and magnitude
(Earth mean equator and equinox of J2000)
```

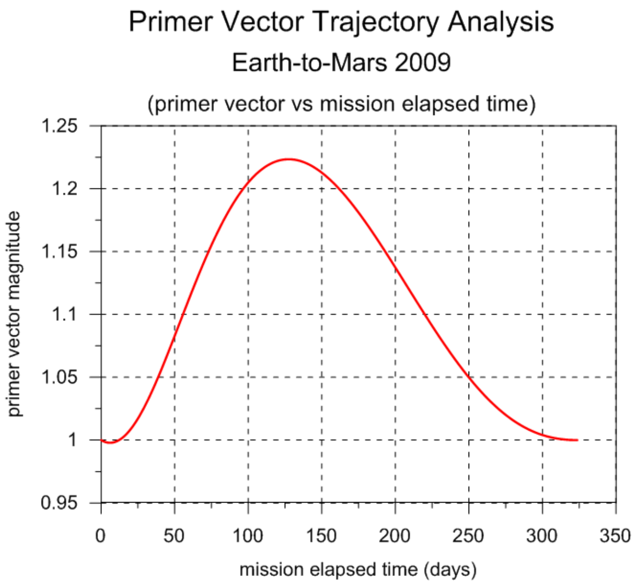
```
----------------------------------------

delta-vx                      2286.93024461405      meters/seconds
delta-vx                      2768.36108454785      meters/seconds
delta-vx                      831.346513595756      meters/seconds

delta-v magnitude             3685.78486401977      meters/seconds
```

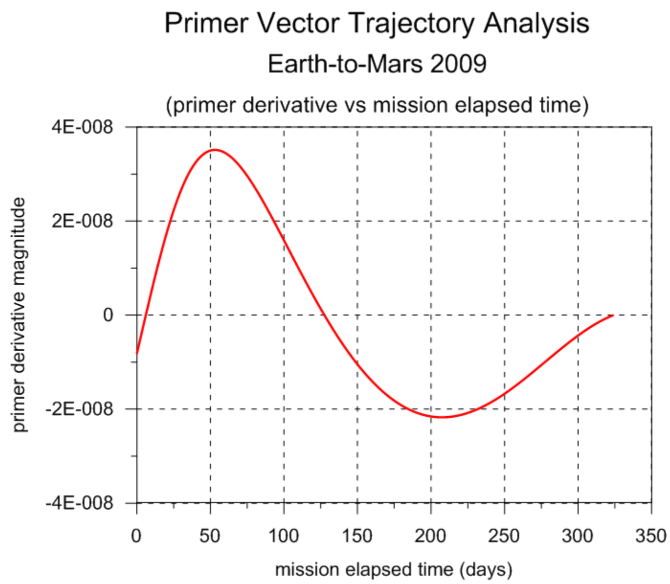
The ipto_ftn software will create a comma-separated-variable (csv) output file named ipto.csv. This file contains the heliocentric, ecliptic state vectors of the spacecraft, the departure planet and the destination celestial body. This data file can be used to create graphic displays of the trajectory and other flight characteristics. Appendix A provides additional information about the contents of this file.

The following is the transfer trajectory for this example. It is a view of the trajectory and planetary orbits from the north pole of the ecliptic looking down on the ecliptic plane. The transfer trajectory is green and the scales are in astronomical units.

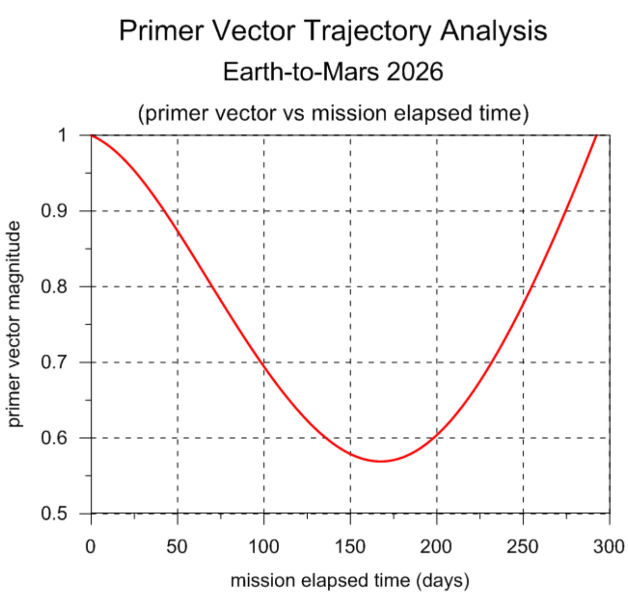


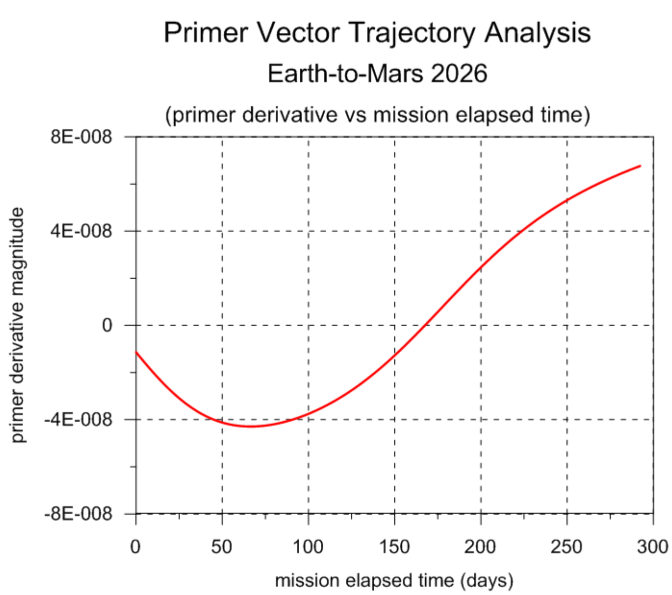
The following are graphic displays of the magnitudes of the primer vector and its derivative for this example. From these two plots we can see that although the solution found by the ipto_ftn software optimizes the total mission delta-v and satisfies the mission constraints, it is not optimal according to primer vector theory which is summarized in the Primer Vector Analysis section.





The following are plots of the primer and its derivative for an Earth-to-Mars mission in 2026.





From the primer vector plot, the solution satisfies the conditions related to the magnitude of the primer vector. However, according to the primer derivative plot there is still room for improvement in the interplanetary trajectory design.

The scalar magnitude of the primer vector derivative at the initial and final impulses provide information about how to improve the nominal transfer trajectory by changing the endpoint times and/or moving the velocity impulse times. The four cases for non-zero slopes are summarized in the Primer Vector Analysis section which can be found on page 14 of this document.

Technical Discussion

This section provides additional details about the software implementation. For good scaling during the optimization, the time unit used in all internal calculations is days, position is expressed in astronomical units, and the unit for velocity and delta-v is astronomical units per day.

An initial guess for the departure and arrival impulsive delta-v vectors can be determined from the solution of the Lambert two-point boundary-value problem (TPBVP). Lambert's Theorem states that the time to traverse a trajectory depends only upon the length of the semimajor axis a of the transfer trajectory, the sum $r_i + r_f$ of the distances of the initial and final positions relative to a central body, and the length c of the chord joining these two positions.

The algorithm used in this computer program to solve the two-body Lambert problem is based on the method described in "A Procedure for the Solution of Lambert's Orbital Boundary-Value Problem" by R. H. Gooding, *Celestial Mechanics and Dynamical Astronomy* **48:** 145-165, 1990. This iterative solution is valid for elliptic, parabolic and hyperbolic transfer orbits which may be either posigrade or retrograde, and involve one or more revolutions about the central body.

The Lambert solution that initializes the `ipto_ftn` computer program uses the user's initial guess for departure and arrival dates.

The $\Delta V's$ required at departure and arrival are simply the differences between the velocity on the optimal transfer trajectory and the heliocentric velocities of the two celestial objects. If we treat each planet as a point mass and assume *impulsive* maneuvers, the *body-centered* magnitude and direction of the required maneuvers are given by the two vector equations:

$$\Delta \mathbf{V}_L = \mathbf{V}_{T_L} - \mathbf{V}_{P_L}$$

$$\Delta \mathbf{V}_A = \mathbf{V}_{P_A} - \mathbf{V}_{T_A}$$

where

$\mathbf{V}_{T_L}$ = heliocentric velocity vector of the transfer trajectory at departure

$\mathbf{V}_{T_A}$ = heliocentric velocity vector of the transfer trajectory at arrival

$\mathbf{V}_{P_L}$ = heliocentric velocity vector of the departure body

$\mathbf{V}_{P_A}$ = heliocentric velocity vector of the arrival body

The scalar magnitude of each maneuver is also called the "hyperbolic excess velocity" or V_∞ at departure and arrival. The hyperbolic excess velocity is the speed of the spacecraft relative to each planet or celestial body at an *infinite* distance from the planet. Furthermore, the *energy* or C_3 at departure or arrival is equal to V_∞^2 for the respective maneuver. C_3 is also equal to twice the orbital energy per unit mass (the specific orbital energy).

The orientation of the departure and arrival hyperbolas is specified in terms of the right ascension (RLA) and declination (DLA) of the outgoing asymptote. These coordinates can be calculated using the Cartesian components of the corresponding V_∞ velocity vector.

The right ascension of the asymptote is determined from

$$\alpha = \tan^{-1}\left(\Delta V_y, \Delta V_z\right)$$

and the geocentric declination of the asymptote is given by

$$\delta = 90^0 - \cos^{-1}\left(\Delta \hat{V}_z\right)$$

where $\Delta \hat{V}_x$, $\Delta \hat{V}_y$ and $\Delta \hat{V}_z$ are the x, y and z components of the unit ΔV vector. The right ascension is computed using a four quadrant inverse tangent function.

In this computer program the heliocentric planetary coordinates and therefore the ΔV vectors are computed in the Earth mean ecliptic and equinox of J2000 coordinate system. In order to determine the orientation of the departure and arrival hyperbolas, these ΔV vectors must be transformed to the Earth mean equator and equinox of J2000 (EME2000) coordinate frame.

The required transformation is given by

$$\Delta \mathbf{V}_{eq} = \begin{bmatrix} 1 & -0.000000479966 & 0 \\ 0.000000440360 & 0.917482137087 & 0.397776982902 \\ -0.000000190919 & -0.397776982902 & 0.917482137087 \end{bmatrix} \Delta \mathbf{V}_{ec}$$

where $\Delta \mathbf{V}_{ec}$ is the delta-velocity vector in the ecliptic frame, and $\Delta \mathbf{V}_{eq}$ is the delta-velocity vector in the equatorial frame.

The transformation of vectors from the equatorial to the ecliptic system involves the transpose of this matrix according to

$$\Delta \mathbf{V}_{ec} = \begin{bmatrix} 1 & -0.000000479966 & 0 \\ 0.000000440360 & 0.917482137087 & 0.397776982902 \\ -0.000000190919 & -0.397776982902 & 0.917482137087 \end{bmatrix}^T \Delta \mathbf{V}_{eq}$$

The `ipto_ftn` software models the planetary coordinates using the DE421 model from JPL.

The following figure illustrates the geometry of the EME2000 coordinate system. The origin of this Earth-centered-inertial (ECI) inertial coordinate system is the geocenter and the fundamental plane is the Earth's mean equator. The z-axis of this system is normal to the Earth's mean equator at epoch J2000, the x-axis is parallel to the vernal equinox of the Earth's mean orbit at epoch J2000, and the y-axis completes the right-handed coordinate system. The epoch J2000 is the Julian Date 2451545.0 which corresponds to January 1, 2000, 12 hours Terrestrial Time (TT).

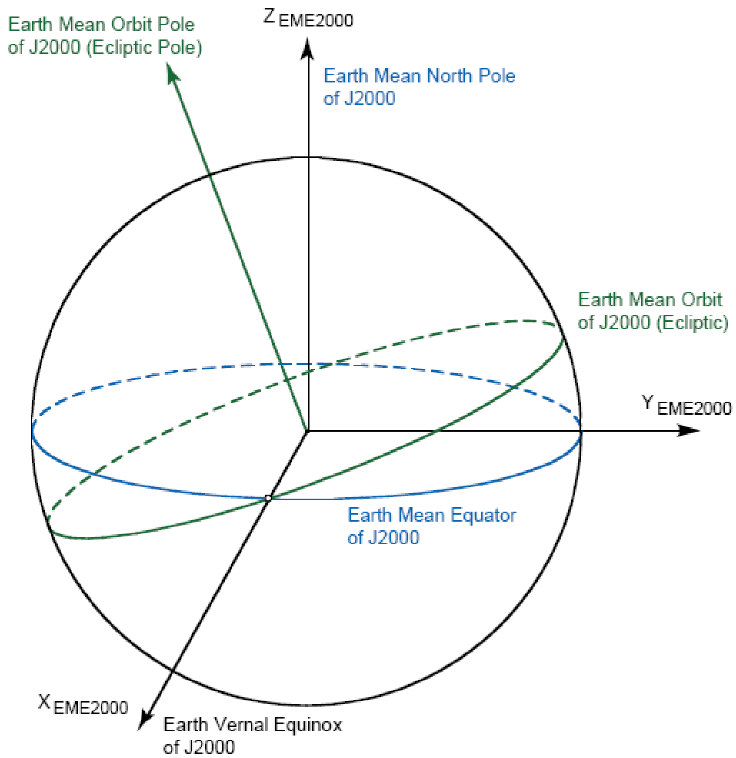


Figure 1. Earth mean equator and equinox of J2000 coordinate system

Important Note

The binary ephemeris file provided with this computer program was created for use on Windows compatible computers. For other platforms, you will need to create or obtain binary files specific to that system. Information and computer programs for creating these files can be found at the JPL solar system FTP site located at ftp://ssd.jpl.nasa.gov/pub/eph/export/. This site provides ASCII data files and Fortran computer programs for creating a binary file. A program for testing the user's ephemeris is also provided along with documentation.

In order to model ballistic interplanetary missions involving asteroids and comets, the classical orbital elements of an asteroid or comet relative to the mean ecliptic and equinox of J2000 coordinate system must be provided by the user. These elements can be obtained from the JPL Near Earth Object (NEO) website (http://neo.jpl.nasa.gov).

These orbital elements consist of the following items:

- TDB calendar date of perihelion passage
- perihelion distance (AU)
- orbital eccentricity (non-dimensional)
- orbital inclination (degrees)
- argument of perihelion (degrees)
- longitude of ascending node (degrees)

The software determines the mean anomaly of the asteroid or comet at any simulation time using the following equation:

$$M = \sqrt{\frac{\mu_s}{a^3}}\, t_{pp} = \sqrt{\frac{\mu_s}{a^3}}\left(JD - JD_{pp}\right)$$

where μ_s is the gravitational constant of the sun, *a* is the semimajor axis of the celestial body's heliocentric orbit, and t_{pp} is the time since perihelion passage.

The semimajor axis is determined from the perihelion distance r_p and orbital eccentricity *e* according to $a = r_p / (1 - e)$.

This solution of Kepler's equation in this computer program is based on a numerical solution devised by Professor J.M.A. Danby at North Carolina State University. Additional information about this algorithm can be found in "The Solution of Kepler's Equation", *Celestial Mechanics*, **31** (1983) 95-107, 317-328 and **40** (1987) 303-312.

The initial guess for Danby's method for elliptic orbits is

$$E_0 = M + 0.85 \operatorname{sign}(\sin M) e$$

The fundamental transcendental equation we want to solve is

$$f(E) = E - e \sin E - M = 0$$

which has the first three derivatives given by

$$f'(E) = 1 - e\cos E \qquad f''(E) = e \sin E \qquad f'''(E) = e\cos E$$

The iteration for an updated eccentric anomaly based on a current value E_n is given by the next four equations:

$$\Delta(E_n) = -\frac{f}{f'}$$

$$\Delta^*(E_n) = -\frac{f}{f' + \frac{1}{2}\Delta f''}$$

$$\Delta_n(E_n) = -\frac{f}{f' + \frac{1}{2}\Delta f'' + \frac{1}{6}\Delta^{*2} f'''}$$

$$E_{n+1} = E_n + \Delta_n$$

This algorithm provides quartic convergence of Kepler's equation. This process is repeated until the convergence test, $\left| f(E) \le \varepsilon \right|$ involving the fundamental equation is satisfied. This tolerance is hardwired in the software to ε = `1.0e-10`.

Finally, the true anomaly of the celestial body can be calculated with the following two equations

$$\sin\theta = \sqrt{1-e^2}\sin E \qquad \cos\theta = \cos E - e$$

and the four quadrant inverse tangent given by

$$\theta = \tan^{-1}\left(\sin\theta, \cos\theta\right)$$

If the orbit is hyperbolic, the initial guess is

$$H_0 = \log\left(\frac{2M}{e} + 1.8\right)$$

where H_0 is the hyperbolic anomaly. The fundamental equation and first three derivatives for this case are as follows:

$$f\left(H\right) = e\sinh H - H - M$$

$$f'\left(H\right) = e\cosh H - 1$$

$$f''\left(H\right) = e\sinh H$$

$$f'''\left(H\right) = e\cosh H$$

Otherwise, the iteration loop which calculates Δ, Δ^*, and so forth is the same. The true anomaly for hyperbolic orbits is determined with this next set of equations:

$$\sin\theta = \sqrt{e^2 - 1}\sinh H \qquad \cos\theta = e - \cosh H$$

The true anomaly is then determined from a four quadrant inverse tangent evaluation of these last two equations. Please note that this algorithm implements a two-body or Keplerian motion of the body.

Departure trajectory characteristics

The algorithm used to compute the trajectory characteristics of the geocentric park orbit and injection hyperbola is described in Appendix C.

Primer Vector Analysis

This section summarizes the primer vector analysis performed by `ipto_ftn` software. The term primer vector was invented by Derek F. Lawden and represents the adjoint vector for velocity. A technical discussion about primer theory can be found in Lawden's classic text, *Optimal Trajectories for Space Navigation*, Butterworths, London, 1963. Another excellent resource is "Primer Vector Theory and Applications", Donald J. Jezewski, NASA TR R-454, November 1975, along with "Optimal, Multi-burn, Space Trajectories", also by Jezewski.

As shown by Lawden, the following four necessary conditions must be satisfied in order for an impulsive orbital transfer to be *locally optimal*:

(1) the primer vector and its first derivative are everywhere continuous

(2) whenever a velocity impulse occurs, the primer is a unit vector aligned with the impulse and has unit magnitude $\left(\mathbf{p} = \hat{\mathbf{p}} = \hat{\mathbf{u}}_T \text{ and } \|\mathbf{p}\| = 1\right)$

(3) the magnitude of the primer vector may not exceed unity on a coasting arc $\left(\|\mathbf{p}\| = p \le 1\right)$

(4) at all interior impulses (not at the initial or final times) $\mathbf{p} \bullet \dot{\mathbf{p}} = 0$; therefore, $d\|\mathbf{p}\|/dt = 0$ at the intermediate impulses

Furthermore, the scalar magnitude of the primer vector derivative at the initial and final impulses provide information about how to improve the nominal transfer trajectory by changing the endpoint times and/or moving the velocity impulse times. These four cases for non-zero slopes are summarized as follows;

- If $\dot{p}_0 > 0$ and $\dot{p}_f < 0 \rightarrow$ perform an initial coast before the first impulse and add a final coast after the second impulse
- If $\dot{p}_0 > 0$ and $\dot{p}_f > 0 \rightarrow$ perform an initial coast before the first impulse and move the second impulse to a later time
- If $\dot{p}_0 < 0$ and $\dot{p}_f < 0 \rightarrow$ perform the first impulse at an earlier time and add a final coast after the second impulse
- If $\dot{p}_0 < 0$ and $\dot{p}_f > 0 \rightarrow$ perform the first impulse at an earlier time and move the second impulse to a later time

The primer vector analysis of a two impulse orbital transfer involves the following steps.
First partition the two-body state transition matrix $\Phi\left(t, t_0\right)$ as follows:

$$\Phi\left(t, t_0\right) = \begin{bmatrix} \dfrac{\partial \mathbf{r}}{\partial \mathbf{r}_0} & \dfrac{\partial \mathbf{r}}{\partial \mathbf{v}_0} \\ \dfrac{\partial \mathbf{v}}{\partial \mathbf{r}_0} & \dfrac{\partial \mathbf{v}}{\partial \mathbf{v}_0} \end{bmatrix} = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} = \begin{bmatrix} \Phi_{rr} & \Phi_{rv} \\ \Phi_{vr} & \Phi_{vv} \end{bmatrix}$$

where

$$\Phi_{11} = \left[\frac{\partial \mathbf{r}}{\partial \mathbf{r}_0}\right] = \begin{bmatrix} \partial x / \partial x_0 & \partial x / \partial y_0 & \partial x / \partial z_0 \\ \partial y / \partial x_0 & \partial y / \partial y_0 & \partial y / \partial z_0 \\ \partial z / \partial x_0 & \partial z / \partial y_0 & \partial z / \partial z_0 \end{bmatrix}$$

and so forth.

The value of the primer vector at any time *t* along a two body trajectory is given by

$$\mathbf{p}(t) = \Phi_{11}(t,t_0)\mathbf{p}_0 + \Phi_{12}(t,t_0)\dot{\mathbf{p}}_0$$

and the value of the primer vector derivative is

$$\dot{\mathbf{p}}(t) = \Phi_{21}(t,t_0)\mathbf{p}_0 + \Phi_{22}(t,t_0)\dot{\mathbf{p}}_0$$

which can also be expressed as

$$\begin{Bmatrix} \mathbf{p} \\ \dot{\mathbf{p}} \end{Bmatrix} = \Phi(t,t_0)\begin{Bmatrix} \mathbf{p}_0 \\ \dot{\mathbf{p}}_0 \end{Bmatrix}$$

The primer vector boundary conditions at the initial and final impulses are as follows:

$$\mathbf{p}(t_0) = \mathbf{p}_0 = \frac{\Delta\mathbf{V}_0}{|\Delta\mathbf{V}_0|}$$

$$\mathbf{p}(t_f) = \mathbf{p}_f = \frac{\Delta\mathbf{V}_f}{|\Delta\mathbf{V}_f|}$$

These two conditions illustrate that at the locations of velocity impulses, the primer vector is a unit vector in the direction of the velocity impulses.

The value of the primer vector derivative at the initial time is

$$\dot{\mathbf{p}}(t_0) = \dot{\mathbf{p}}_0 = \Phi_{12}^{-1}(t_f,t_0)\left\{\mathbf{p}_f - \Phi_{11}(t_f,t_0)\mathbf{p}_0\right\}$$

provided the Φ_{12} sub-matrix is non-singular.

The scalar magnitude of the derivative of the primer vector can be determined from

$$\frac{d\|\mathbf{p}\|}{dt} = \frac{d}{dt}(\mathbf{p}\bullet\mathbf{p})^2 = \frac{\dot{\mathbf{p}}\bullet\mathbf{p}}{\|\mathbf{p}\|}$$

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"Optimal Interplanetary Orbit Transfers by Direct Transcription", John T. Betts, *The Journal of the Astronautical Sciences*, Vol. 42, No. 3, July-September 1994, pp. 247-268.

APPENDIX A

Contents of the Simulation Summary and CSV File

This appendix is a brief summary of the information contained in the simulation summary screen displays and the CSV data file produced by the `ipto_ftn` software. All output except the coordinates of the departure hyperbola is computed and displayed in a heliocentric, Earth mean ecliptic and equinox of J2000 coordinate system. The time system is Barycentric Dynamical Time (TDB).

The simulation summary screen display contains the following information:

calendar date = calendar date of trajectory event

TDB time = TDB time of trajectory event

TDB julian date = TDB julian date of trajectory event

sma (au) = semimajor axis in astronomical units

eccentricity = orbital eccentricity (non-dimensional)

inclination (deg) = orbital inclination in degrees

argper (deg) = argument of perigee in degrees

raan (deg) = right ascension of the ascending node in degrees

true anomaly (deg) = true anomaly in degrees

arglat (deg) = argument of latitude in degrees. The argument of latitude is the sum of true anomaly and argument of perigee.

period (days) = orbital period in days

rx (km) = x-component of the spacecraft's heliocentric position vector in kilometers

ry (km) = y-component of the spacecraft's heliocentric position vector in kilometers

rz (km) = z-component of the spacecraft's heliocentric position vector in kilometers

rmag (km) = scalar magnitude of the spacecraft's heliocentric position vector in kilometers

vx (kps) = x-component of the spacecraft's heliocentric velocity vector in kilometers per second

vy (kps) = y-component of the spacecraft's heliocentric velocity vector in kilometers per second

vz (ksp) = z-component of the spacecraft's heliocentric velocity vector in kilometers per second

vmag (kps) = scalar magnitude of the spacecraft's heliocentric velocity vector in kilometers per second

c3 = specific orbital energy in kilometers**2/second**2

v-infinity = speed of the spacecraft at infinity in kilometers/second

rasc-asymptote = right ascension of the departure/arrival hyperbola in degrees

decl-asymptote = declination of the departure/arrival hyperbola in degrees

time of flight = departure to arrival flight time in days

The comma-separated-variable disk file (`ipto.csv`) contains the following information:

time (days) = simulation time since departure in days

rs2sc-x (au) = x-component of the spacecraft's heliocentric position vector in astronomical units

rs2sc-y (au) = y-component of the spacecraft's heliocentric position vector in astronomical units

rs2sc-z (au) = z-component of the spacecraft's heliocentric position vector in astronomical units

rs2sc-mag (au) = the spacecraft's heliocentric radius in astronomical units

vs2sc-y (km/sec) = y-component of the spacecraft's heliocentric velocity vector in astronomical units per day

vs2sc-z (km/sec) = z-component of the spacecraft's heliocentric position vector in astronomical units per day

vs2sc-mag (km/sec) = the spacecraft's heliocentric speed in astronomical units per day

rs2p1-x (au) = x-component of the departure planet's heliocentric position vector in astronomical units

rs2p1-y (au) = y-component of the departure planet's heliocentric position vector in astronomical units

rs2p1-z (au) = z-component of the departure planet's heliocentric position vector in astronomical units

rs2p1-mag (au) = departure planet heliocentric radius in astronomical units

vs2p1-x (au) = x-component of the departure planet's heliocentric velocity vector in astronomical units per day

vs2p1-y (au) = y-component of the departure planet's heliocentric velocity vector in astronomical units per day

vs2p1-z (au) = z-component of the departure planet's heliocentric velocity vector in astronomical units per day

vs2p1-mag (au) = departure planet's heliocentric speed in astronomical units per day

rs2b2-x (au) = x-component of the arrival body's heliocentric position vector in astronomical units

rs2b2-y (au) = y-component of the arrival body's heliocentric position vector in astronomical units

rs2b2-z (au) = z-component of the arrival body's heliocentric position vector in astronomical units

rs2b2-mag (au) = flyby planet heliocentric radius in astronomical units

vs2b2-x (au) = x-component of the arrival body's heliocentric velocity vector in astronomical units per day

vs2b2-y (au) = y-component of the arrival body's heliocentric velocity vector in astronomical units per day

vs2b2-z (au) = z-component of the arrival body's heliocentric velocity vector in astronomical units per day

vs2b2-mag (au) = flyby planet heliocentric speed in astronomical units per day

sma-heo (au) = heliocentric semimajor axis of the spacecraft in astronomical units

ecc-heo = heliocentric orbital eccentricity of the spacecraft(non-dimensional)

inc-heo (deg) = heliocentric orbital inclination of the spacecraft in degrees

argper-heo (deg) = heliocentric argument of perigee of the spacecraft in degrees

raan-heo (deg) = heliocentric right ascension of the ascending node of the spacecraft in degrees

tanom-heo (deg) = heliocentric true anomaly of the spacecraft in degrees

The comma-separated-variable disk file (`primer.csv`) contains the mission elapsed time in column 1, the magnitude of the primer vector in column 2, and the primer derivative magnitude in column 3. This file is used by the `pplot.exe` executable program to interactively plot the primer behavior after the trajectory optimization is complete. The `pplot` program requires the `dislin.dll` run-time file.

APPENDIX B

Earth-to-Tempel 1 Trajectory Analysis

This appendix summarizes typical trajectory characteristics of a ballistic and patched-conic mission from Earth to the comet Tempel 1. This simulation example minimizes the magnitude of the departure delta-v at the Earth departure.

Here's the `ipto_ftn` input data file for this example.

```
*****************************************
** patched-conic ballistic interplanetary
** trajectory design and optimization
** Earth-to-Tempel1 example - tempel1.in
** July 12, 2011
*****************************************

***************
simulation type
***************
 1 = minimize departure delta-v
 2 = minimize arrival delta-v
 3 = minimize total delta-v
 4 = no optimization
--------------------
1

departure calendar date initial guess (month, day, year)
12, 1, 2004

departure date search boundary (days)
-60, +60

arrival calendar date initial guess (month, day, year)
7, 1, 2005

arrival date search boundary (days)
-90, +90

********************
* departure planet *
********************
 1 = Mercury
 2 = Venus
 3 = Earth
 4 = Mars
 5 = Jupiter
 6 = Saturn
 7 = Uranus
 8 = Neptune
 9 = Pluto
----------
3

**************************
* arrival celestial body *
**************************
 1 = Mercury
 2 = Venus
 3 = Earth
 4 = Mars
 5 = Jupiter
 6 = Saturn
 7 = Uranus
 8 = Neptune
 9 = Pluto
 0 = asteroid/comet
-------------------
0
```

```
**********************************
* asteroid/comet orbital elements *
* (heliocentric, ecliptic J2000)  *
**********************************

asteroid/comet name
Tempel 1

calendar date of perihelion passage (month, day, year)
7, 5.3153, 2005

perihelion distance (au)
1.506167

orbital eccentricity (nd)
0.517491

orbital inclination (degrees)
10.5301

argument of perihelion (degrees)
178.8390

longitude of the ascending node (degrees)
68.9734

******************************
geocentric trajectory modeling
******************************

circular park orbit altitude (kilometers)
185.32

circular park orbit inclination (degrees)
28.5
```

Here’s the optimal solution for this example.

```
program ipto_ftn - interplanetary mission design
================================================

input data file ==> tempel1.in

minimize departure delta-v


heliocentric coordinates of the planet at departure
(Earth mean ecliptic and equinox of J2000)
------------------------------------------

calendar date           January 10, 2005

TDB time                08:46:27.148

TDB Julian date         2453380.86559199

     sma (au)            eccentricity       inclination (deg)       argper (deg)
 0.100096582507D+01    0.176105214436D-01    0.774607945293D-03    0.317595688360D+03

    raan (deg)         true anomaly (deg)      arglat (deg)         period (days)
 0.145608604412D+03    0.694563012791D+01    0.324541318488D+03    0.365786187478D+03

     rx (km)                ry (km)               rz (km)              rmag (km)
 -.506809540009D+08    0.138119195870D+09    -.115387155096D+04    0.147123999971D+09

     vx (kps)              vy (kps)              vz (kps)              vmag (kps)
 -.284630343795D+02    -.103765794905D+02    0.333117906408D-03    0.302955067316D+02


spacecraft heliocentric coordinates after the first impulse
(Earth mean ecliptic and equinox of J2000)
------------------------------------------
```

```
calendar date           January 10, 2005

TDB time                08:46:27.148

TDB Julian date         2453380.86559199

     sma (au)            eccentricity      inclination (deg)      argper (deg)
 0.130073947041D+01    0.243921682908D+00    0.572780597816D+00    0.180323920364D+03

    raan (deg)         true anomaly (deg)      arglat (deg)        period (days)
 0.290104974270D+03    0.359721030514D+03    0.180044950879D+03    0.541856023177D+03

     rx (km)                ry (km)               rz (km)              rmag (km)
 -.506809540009D+08    0.138119195870D+09    -.115387155096D+04    0.147123999971D+09

     vx (kps)              vy (kps)               vz (kps)             vmag (kps)
 -.314345096795D+02    -.115685338523D+02    -.334863677725D+00    0.334973358339D+02


spacecraft heliocentric coordinates prior to the second impulse
(Earth mean ecliptic and equinox of J2000)
------------------------------------------

calendar date           July 10, 2005

TDB time                02:23:55.211

TDB Julian date         2453561.59994457

     sma (au)            eccentricity      inclination (deg)      argper (deg)
 0.130073947041D+01    0.243921682908D+00    0.572780597816D+00    0.180323920364D+03

    raan (deg)         true anomaly (deg)      arglat (deg)        period (days)
 0.290104974270D+03    0.140490429312D+03    0.320814349677D+03    0.541856023177D+03

     rx (km)                ry (km)               rz (km)              rmag (km)
 -.736878055673D+08    -.213046898675D+09    -.142391291678D+07    0.225434915780D+09

     vx (kps)              vy (kps)               vz (kps)             vmag (kps)
 0.192932872867D+02    -.110959812622D+02    0.142995460818D+00    0.222569580626D+02


spacecraft heliocentric coordinates after the second impulse
(Earth mean ecliptic and equinox of J2000)
------------------------------------------

calendar date           July 10, 2005

TDB time                02:23:55.211

TDB Julian date         2453561.59994457

     sma (au)            eccentricity      inclination (deg)      argper (deg)
 0.312153141185D+01    0.517491000000D+00    0.105301000000D+02    0.178839000000D+03

    raan (deg)         true anomaly (deg)      arglat (deg)        period (days)
 0.689734000000D+02    0.314165635128D+01    0.181980656351D+03    0.201441984506D+04

     rx (km)                ry (km)               rz (km)              rmag (km)
 -.736878055673D+08    -.213046898675D+09    -.142391291678D+07    0.225434915780D+09

     vx (kps)              vy (kps)               vz (kps)             vmag (kps)
 0.275932747334D+02    -.100985870885D+02    -.546110371277D+01    0.298863501528D+02


heliocentric coordinates of celestial body at arrival
(Earth mean ecliptic and equinox of J2000)
------------------------------------------

calendar date           July 10, 2005

TDB time                02:23:55.211
```

```
TDB Julian date         2453561.59994457

     sma (au)            eccentricity      inclination (deg)      argper (deg)
 0.312153141185D+01    0.517491000000D+00    0.105301000000D+02    0.178839000000D+03

    raan (deg)         true anomaly (deg)       arglat (deg)         period (days)
 0.689734000000D+02    0.314165635128D+01    0.181980656351D+03    0.201441984506D+04

     rx (km)               ry (km)               rz (km)              rmag (km)
 -.736878055674D+08    -.213046898675D+09    -.142391291678D+07    0.225434915780D+09
     vx (kps)              vy (kps)              vz (kps)             vmag (kps)
 0.275932747334D+02    -.100985870885D+02    -.546110371277D+01    0.298863501528D+02
```

IMPULSIVE MANEUVER SUMMARY
----

departure heliocentric delta-v vector and magnitude
(Earth mean ecliptic and equinox of J2000)

```
x-component of delta-v    -2971.47529998509      meters/second
y-component of delta-v    -1191.95436183438      meters/second
z-component of delta-v    -335.196795631003      meters/second

delta-v magnitude          3219.12683051146      meters/second
```

arrival heliocentric delta-v vector and magnitude
(Earth mean ecliptic and equinox of J2000)

```
x-component of delta-v     8299.98744662347      meters/second
y-component of delta-v     997.394173632673      meters/second
z-component of delta-v    -5604.09917358654      meters/second

delta-v magnitude          10064.3188691087      meters/second
```

HYPERBOLIC TRAJECTORY CHARACTERISTICS
(Earth mean equator and equinox of J2000)
----

departure hyperbola

```
c3                   10.3627775509188      km**2/sec**2

v-infinity           3219.12683051146      meters/second

decl-asymptote      -14.0530519629276      degrees

rasc-asymptote       197.908752800624      degrees
```

arrival hyperbola

```
c3                   101.290514299097      km**2/sec**2

v-infinity           10064.3188691087      meters/second

decl-asymptote      -28.1290885818470      degrees

rasc-asymptote       20.7480954068751      degrees


time of flight       180.734352584928      days
```

orbital elements and state vector of park orbit at injection - opportunity #1
(Earth mean equator and equinox of J2000)

```
     sma (km)          eccentricity       inclination (deg)     argper (deg)
 0.6563460000D+04    0.0000000000D+00    0.2850000000D+02    0.0000000000D+00

    raan (deg)      true anomaly (deg)     arglat (deg)       period (min)
 0.3504560109D+03    0.6191429730D+02    0.6191429730D+02    0.8819805229D+02

     rx (km)             ry (km)             rz (km)            rmag (km)
 0.3891009354D+04    0.4506079485D+04    0.2763023898D+04    0.6563460000D+04

     vx (kps)            vy (kps)            vz (kps)           vmag (kps)
 -.6245534232D+01    0.4319586779D+01    0.1750629357D+01    0.7792960344D+01
```

```
orbital elements and state vector of hyperbola at injection - opportunity #1
(Earth mean equator and equinox of J2000)
-----------------------------------------

     sma (km)          eccentricity       inclination (deg)     argper (deg)
 -.3846463359D+05    0.1170636228D+01    0.2850000000D+02    0.6191429730D+02

    raan (deg)      true anomaly (deg)     arglat (deg)       period (min)
 0.3504560109D+03    0.0000000000D+00    0.6191429730D+02    0.1666666667D+98

     rx (km)             ry (km)             rz (km)            rmag (km)
 0.3891009354D+04    0.4506079485D+04    0.2763023898D+04    0.6563460000D+04

     vx (kps)            vy (kps)            vz (kps)           vmag (kps)
 -.9201595051D+01    0.6364081414D+01    0.2579216098D+01    0.1148143020D+02
```

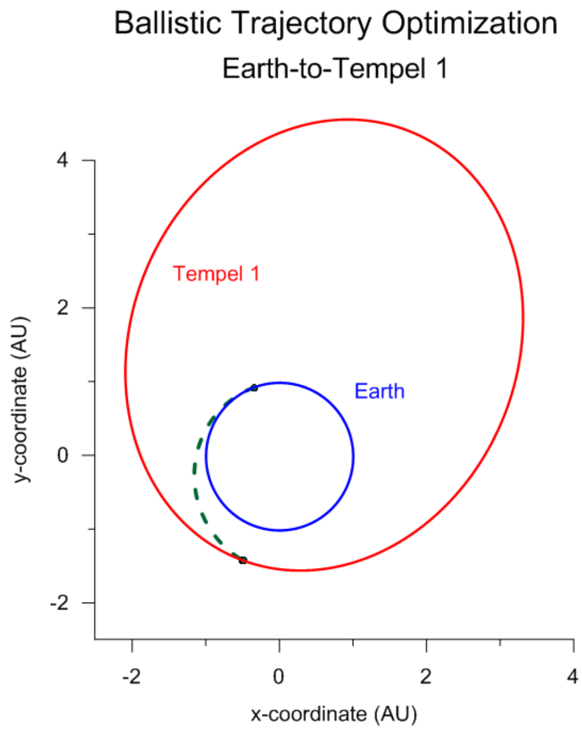
```
injection delta-v vector and magnitude - opportunity #1
(Earth mean equator and equinox of J2000)
-----------------------------------------

x-component of delta-v      -2956.06081922647      meters/second

y-component of delta-v       2044.49463520536      meters/second

z-component of delta-v       828.586740484390      meters/second

delta-v magnitude            3688.46985440520      meters/second
```

```
orbital elements and state vector of park orbit at injection - opportunity #2
(Earth mean equator and equinox of J2000)
-----------------------------------------


     sma (km)          eccentricity       inclination (deg)     argper (deg)
 0.6563460000D+04    0.0000000000D+00    0.2850000000D+02    0.0000000000D+00

    raan (deg)      true anomaly (deg)     arglat (deg)       period (min)
 0.2253614947D+03    0.1807347620D+03    0.1807347620D+03    0.8819805229D+02

     rx (km)             ry (km)             rz (km)            rmag (km)
 0.4558681755D+04    0.4721844438D+04    -.4016130988D+02    0.6563460000D+04

     vx (kps)            vy (kps)            vz (kps)           vmag (kps)
 -.4942955739D+01    0.4740527927D+01    -.3718173539D+01    0.7792960344D+01
```

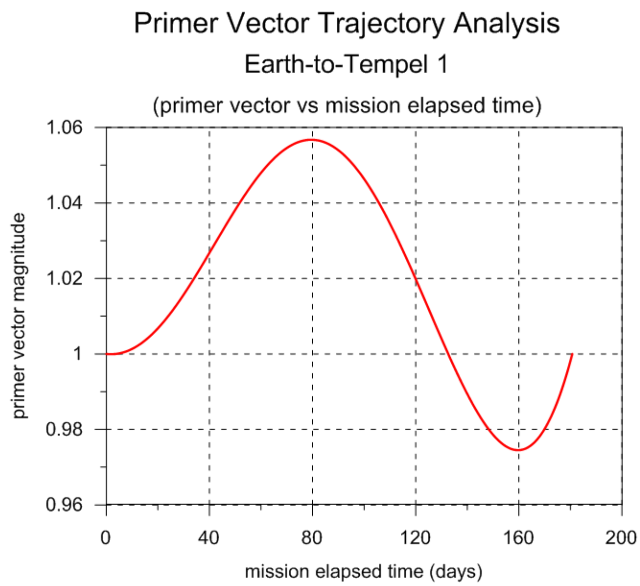
```
orbital elements and state vector of hyperbola at injection - opportunity #2
(Earth mean equator and equinox of J2000)
-----------------------------------------


     sma (km)          eccentricity       inclination (deg)     argper (deg)
 -.3846463359D+05    0.1170636228D+01    0.2850000000D+02    0.1807347620D+03

    raan (deg)      true anomaly (deg)     arglat (deg)       period (min)
 0.2253614947D+03    0.0000000000D+00    0.1807347620D+03    0.1666666667D+98

     rx (km)             ry (km)             rz (km)            rmag (km)
 0.4558681755D+04    0.4721844438D+04    -.4016130988D+02    0.6563460000D+04
```

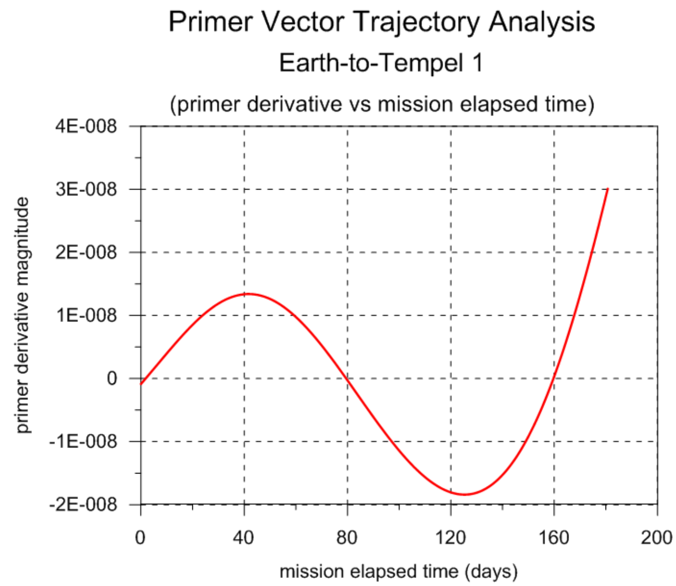
```
      vx (kps)            vy (kps)            vz (kps)           vmag (kps)
 -.7282495840D+01    0.6984257342D+01    -.5478014524D+01    0.1148143020D+02


 injection delta-v vector and magnitude - opportunity #2
 (Earth mean equator and equinox of J2000)
 -----------------------------------------

 x-component of delta-v      -2339.54010141834      meters/second

 y-component of delta-v       2243.72941478197      meters/second

 z-component of delta-v      -1759.84098541703      meters/second

 delta-v magnitude            3688.46985440520      meters/second
```

Here's the graphics display of the interplanetary transfer trajectory along with the heliocentric orbits of the Earth and Tempel 1. We can see from this graphics display and the screen display data above (heliocentric true anomaly $\approx$ 3 degrees) that spacecraft encounter with Tempel 1 occurs near perihelion of the comet's orbit.



The following are graphic displays of the magnitudes of the primer vector and its derivative for this orbit transfer example.





From these two plots we can see that although the solution found by the `ipto_ftn` software optimizes the departure delta-v and satisfies the mission constraints, it is not optimal according to primer vector theory.

APPENDIX C

Interplanetary Injection from a Circular Park Orbit

The algorithm implemented in this scientific simulation assumes that the spacecraft is initially in a circular Earth park orbit. Furthermore, the orbital transfer maneuver is assumed to be impulsive which implies an instantaneous change in velocity but not change in position. In the following discussion, *i* is the orbital inclination of the initial circular Earth park orbit and δ_∞ is the declination of the outgoing or departure hyperbola.

Whenever $i > |\delta_\infty|$, there will be two coplanar opportunities to establish a departure hyperbola that will satisfy the energy and orientation of the outgoing asymptote. One injection opportunity will occur while the spacecraft is ascending and the other while the spacecraft is descending along the park orbit. For coplanar orbital transfer, the impulse is applied at the perigee of the departure hyperbola.

For the case where $|\delta_\infty| > i$, there will be a single *non-coplanar* injection opportunity.

Coplanar Transfer - Orientation of the park orbit and departure hyperbola

This section summarizes the equations used to determine the right ascension of the ascending node (RAAN) of the park orbit and the injection true anomaly on the park orbit.
A unit vector is the direction of the departure asymptote is given by

$$\hat{\mathbf{s}} = \begin{Bmatrix} \cos\delta_\infty \cos\alpha_\infty \\ \cos\delta_\infty \sin\alpha_\infty \\ \sin\delta_\infty \end{Bmatrix}$$

where

$\alpha_\infty =$ right ascension of departure asymptote

$\delta_\infty =$ declination of departure asymptote

The angle between the outgoing asymptote and the spin axis of the Earth is given by

$$\beta = \cos^{-1}\left(\hat{\mathbf{s}} \bullet \hat{\mathbf{z}}\right)$$

where $\hat{\mathbf{z}} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$. Note that $\beta = 90^\circ - \delta_\infty$.

The park orbit right ascension of the ascending node for each opportunity can be determined from

$$\Omega_1 = 180^\circ + \alpha_\infty + \sin^{-1}\left(\frac{\cot\beta}{\tan i}\right)$$

$$\Omega_2 = 360^\circ + \alpha_\infty - \sin^{-1}\left(\frac{\cot\beta}{\tan i}\right)$$

The true anomaly on the park orbit for each injection opportunity can be determined from

$$\theta_1 = \cos^{-1}\left(\frac{\cos\beta}{\sin i}\right) - \eta$$

$$\theta_2 = -\cos^{-1}\left(\frac{\cos\beta}{\sin i}\right) - \eta$$

where

$$\eta = \sin^{-1}\left(\frac{1}{1 + r_p V_\infty^2 / \mu}\right)$$

In the last equation, r_p is the geocentric radius of the park orbit and μ is the gravitational constant of the Earth. The velocity vector at infinity V_∞ is determined from $V_\infty = \sqrt{C_3}$.

For a tangential impulsive injection maneuver that occurs at perigee of the hyperbola, the true anomaly on the hyperbola is zero. Furthermore, since the orbit transfer modeled by this software is coplanar, the right ascension of the ascending node computed above should be the same for both the park orbit and the departure hyperbola. This can be verified by examining the hyperbola's right ascension of the ascending node (RAAN) which is computed using the state vector at injection.

Coplanar Transfer - Departure delta-V

The velocity vector at any geocentric position vector $\mathbf{r}$ required to achieve a departure hyperbola defined by V_∞, α_∞ and δ_∞ is given by

$$\mathbf{v}_h = \left(d + \frac{1}{2}V_\infty\right)\hat{\mathbf{s}} + \left(d - \frac{1}{2}V_\infty\right)\hat{\mathbf{r}}$$

where

$$d = \sqrt{\frac{\mu}{(1+\cos\psi)\, r_p} + \frac{V_\infty^2}{4}}$$

and ψ is the angle between the spacecraft's position vector and the departure asymptote unit vector which can be computed using

$$\cos\psi = \hat{\mathbf{s}} \bullet \hat{\mathbf{r}}$$

The injection $\Delta\mathbf{v}$ vector can be determined from the following expression

$$\Delta\mathbf{v} = \mathbf{v}_h - \mathbf{v}_p$$

where $\mathbf{v}_p$ is the inertial velocity vector in the park orbit prior to injection and $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$.

Finally, the scalar injection delta-v is $\Delta v = |\Delta \mathbf{v}|$. The injection delta-v is also given by

$$\Delta v = \sqrt{2\frac{\mu}{r_p} + V_\infty^2} - \sqrt{\frac{\mu}{r_p}}$$

Non-coplanar Transfer – Park Orbit Orientation and Departure delta-V

A geocentric unit vector is the direction of the departure asymptote is given by

$$\hat{\mathbf{i}}_\infty = \begin{Bmatrix} \cos\delta_\infty \cos\alpha_\infty \\ \cos\delta_\infty \sin\alpha_\infty \\ \sin\delta_\infty \end{Bmatrix}$$

where

$\alpha_\infty =$ right ascension of departure asymptote

$\delta_\infty =$ declination of departure asymptote

The velocity vector of the spacecraft on the initial circular orbit is given by

$$\mathbf{v}_0 = \sqrt{\frac{\mu}{r}}\,\hat{\mathbf{i}}_\theta$$

The velocity vector at any geocentric position vector $\mathbf{r}$ required to achieve a departure hyperbola defined by v_∞, α_∞ and δ_∞ is given by

$$\mathbf{v}_1 = \frac{1}{2} v_\infty \left[(D+1)\hat{\mathbf{i}}_\infty + (D-1)\hat{\mathbf{i}}_r \right]$$

where

$$D = \sqrt{1 + \frac{4\mu}{r v_\infty^2 \left(1 + \hat{\mathbf{i}}_\infty \bullet \hat{\mathbf{i}}_r\right)}}$$

and

$$\hat{\mathbf{i}}_r = \begin{bmatrix} \cos\Omega\cos\theta - \sin\Omega\sin\theta\cos i \\ \sin\Omega\cos\theta + \cos\Omega\sin\theta\cos i \\ \sin\theta\sin i \end{bmatrix}$$

$$\hat{\mathbf{i}}_\theta = \begin{bmatrix} -\cos\Omega\sin\theta - \sin\Omega\cos\theta\cos i \\ -\sin\Omega\sin\theta + \cos\Omega\cos\theta\cos i \\ \cos\theta\sin i \end{bmatrix}$$

In these equations, Ω is the right ascension of the ascending node, i is the orbital inclination, θ is the true anomaly at injection, r is the geocentric radius of the park orbit and $v_\infty = \sqrt{C_3}$.

The injection $\Delta\mathbf{v}$ vector can be determined from the following expression

$$\Delta\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_0$$

Finally, the scalar injection delta-v is $\Delta v = \left|\Delta\mathbf{v}\right|$.

The orientation of the park orbit and departure hyperbola at injection is computed using a two-dimensional grid search involving the park orbit right ascension of the ascending node (RAAN) and the true anomaly of the impulsive maneuver on the park orbit. During the grid search, `ipto_ftn` uses a nonlinear programming (NLP) algorithm to find the current minimum delta-v and saves the RAAN and true anomaly values corresponding to the "best" delta-v.