A MATLAB Script for Creating Interplanetary Pork Chop Plots

This document describes a MATLAB script called `porkchop` that can be used to create and plot interplanetary "pork chop" plots for Type I and Type II Earth-to-Mars missions. These plots illustrate the behavior of launch energy (C3L), right ascension (RLA) and declination (DLA) of the departure hyperbola, time-of-flight, and arrival v-infinity for a range of launch and arrival calendar dates. The data required for these contour plots is created by solving the heliocentric, two-body "patched-conic" Lambert problem. A patched-conic trajectory ignores the gravitational effect of both the launch and arrivals planets on the heliocentric trajectory. Type I trajectories are characterized by heliocentric Earth-to-Mars transfer angles which are less than 180 degrees while Type II trajectories have transfer angles greater than 180 degrees. Pork chop plots are typically used for preliminary mission analysis.

The angular coordinates of the launch hyperbola (DLA and RLA) are provided in the Earth mean equator and equinox of J2000 (EME2000) coordinate system. The `porkchop` script can be easily modified for other departure and arrival planets.

Typical user interaction

The following is typical user interaction with this MATLAB application. This example creates typical data and plots for the Phoenix 2007 mission. In the following discussion the actual inputs are in bold *courier* font and all explanations are in *times italic* font.

```
program porkchop

< interplanetary pork chop plots >
```

The first user input is the nominal launch calendar date in the order month, day, and year. Please be sure to include all four digits of the calendar year.

```
nominal launch date

please input the calendar date
(1 <= month <= 12, 1 <= day <= 31, year = all digits!)
? 9,25,2007
```

The next user input is the calendar date of the nominal arrival at Mars.

```
nominal arrival date

please input the calendar date
(1 <= month <= 12, 1 <= day <= 31, year = all digits!)
? 8,15,2008
```

The next input is the analysis "span" centered about the nominal launch date. This number should be input in days. The software will create data over the range nominal – span and nominal + span.

```
please input the launch date span in days
? 60
```

The next input is the span centered about the nominal arrival date. This number should also be input in days.

```
please input the arrival date span in days
? 180
```

The next input is the step size for the parametric scan. This number should be input in days and need not be an integer.

```
please input the step size in days
? 1
```

The next five inputs are MATLAB vectors that define the contour levels for each flight parameter. To accept default values for the contour levels, the user should input []. Otherwise, the input should be in the format [first level, second level, third level, …, last level].

```
please input the launch energy contour levels in km^2/sec^2 ([] for defaults)
? []
```

The default launch energy contour levels are

[6,7,8,9,10,11,12,13,14,15,20,25,30,35,40,45,50]

```
please input the arrival v-infinity contour levels in km/sec ([] for defaults)
? []
```

The default arrival v-infinity contour levels are

[1.5,1.8,2.0,2.1,2.2,2.3,2.4,2.5,2.6,2.7,2.8,2.9,3.0,3.5,4.0,5.0,6.0,7.0,8.0]

```
please input the launch declination contour levels in degrees ([] for defaults)
? []
```

The default launch declination contour levels are

[-30, -25, -20, -15, -10, -5, 0, 5, 10, 15, 20, 25, 30]

```
please input the launch right ascension contour levels in degrees ([] for defaults)
? []
```

The default launch right ascension contour levels are

[0, 15, 30, 45, 60, 75, 90, 105, 120, 135, 150, 165, 180, 195, 210, 225, 240, 255, 270, 285, 300, 315, 330]

```
please input the time-of-flight contour levels in days ([] for defaults)
? []
```

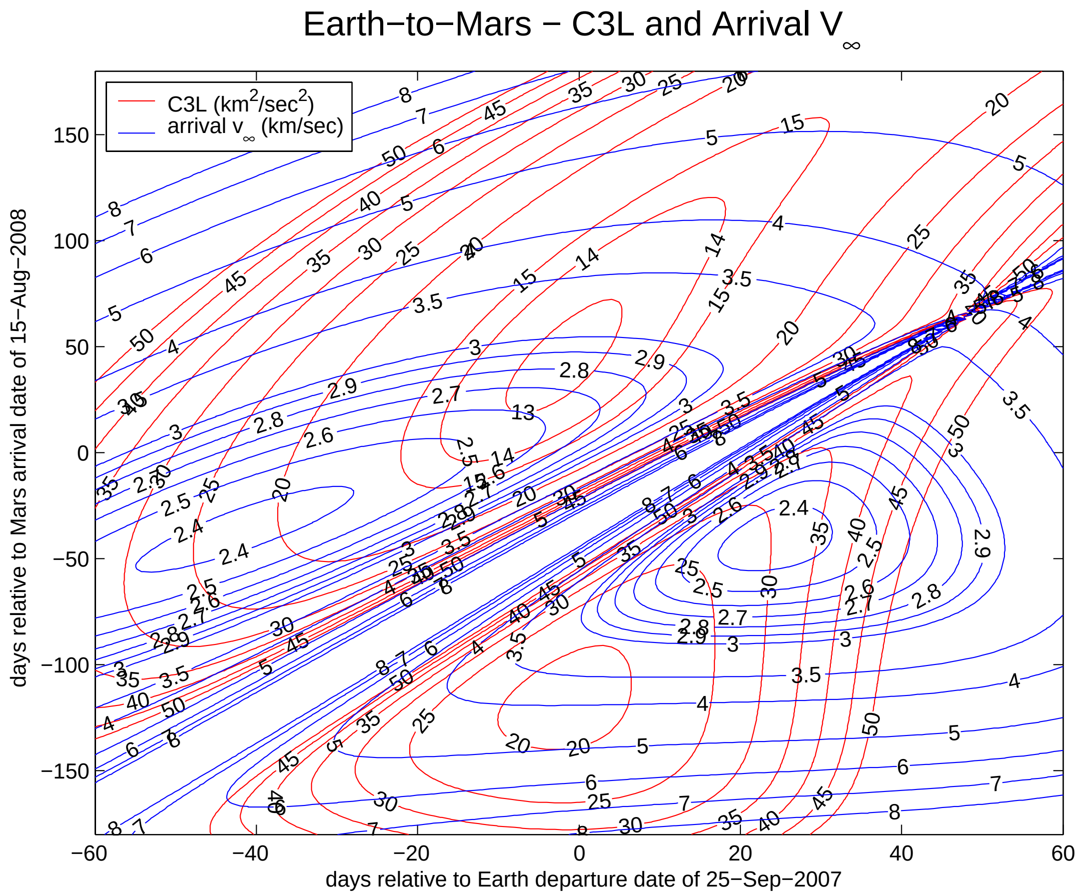
The default time-of-flight contour levels are

[100, 150, 200, 250, 300, 350, 400]

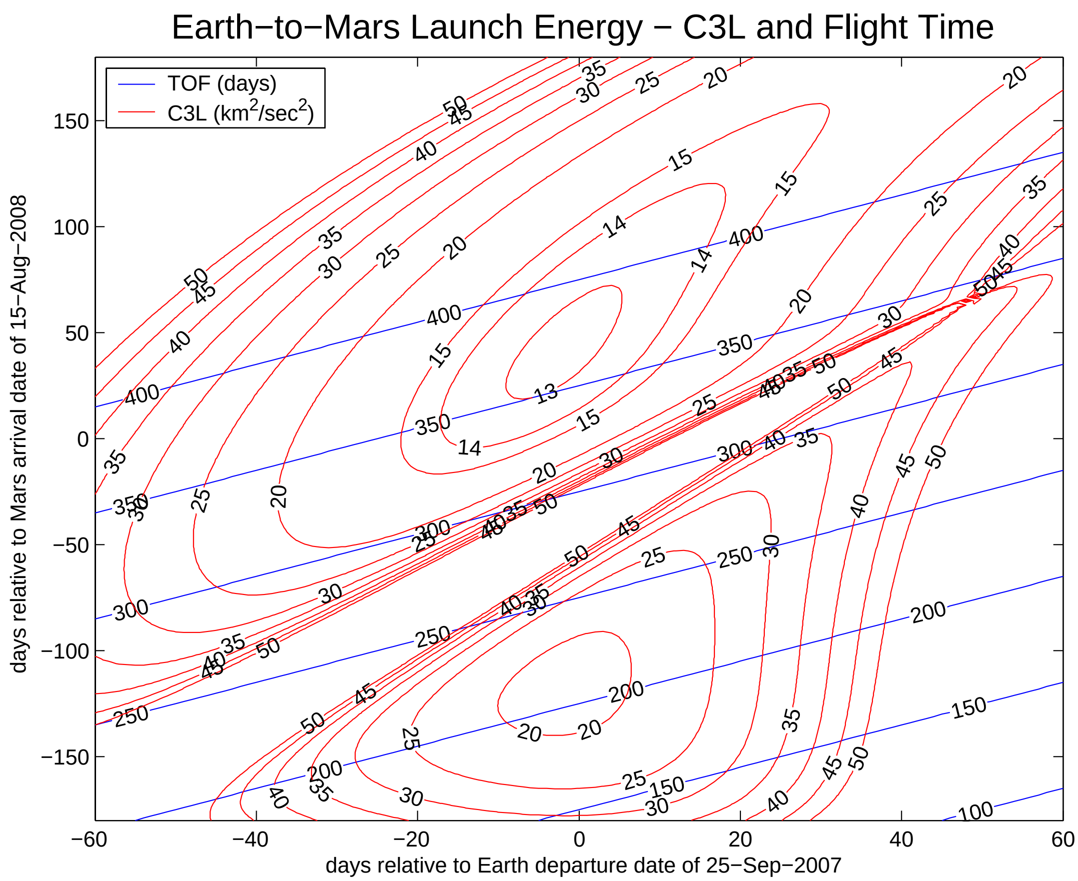
Of course, the user can directly edit the MATLAB source code and change any default values.

Pork chop graphics

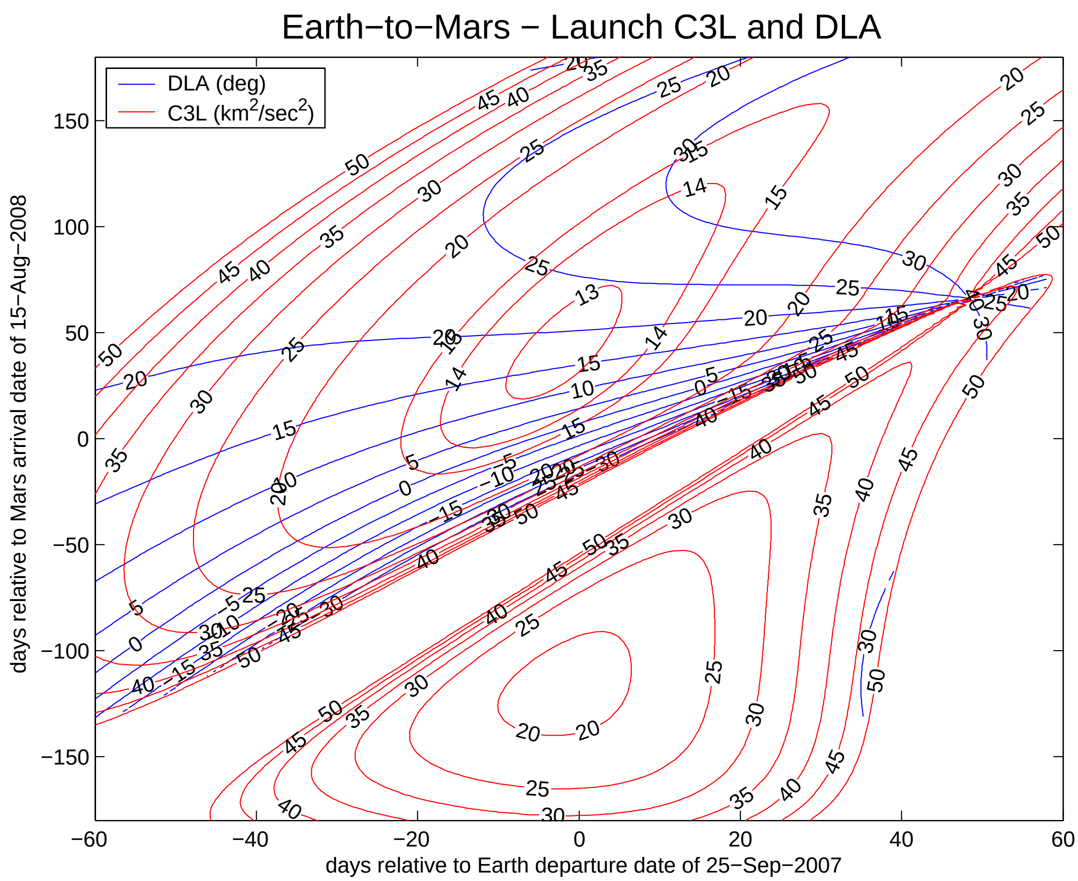
This section contains pork chop plots for this example. The first contour plot summarizes the behavior of the launch energy (C3L) and the arrival v-infinity speed at Mars.



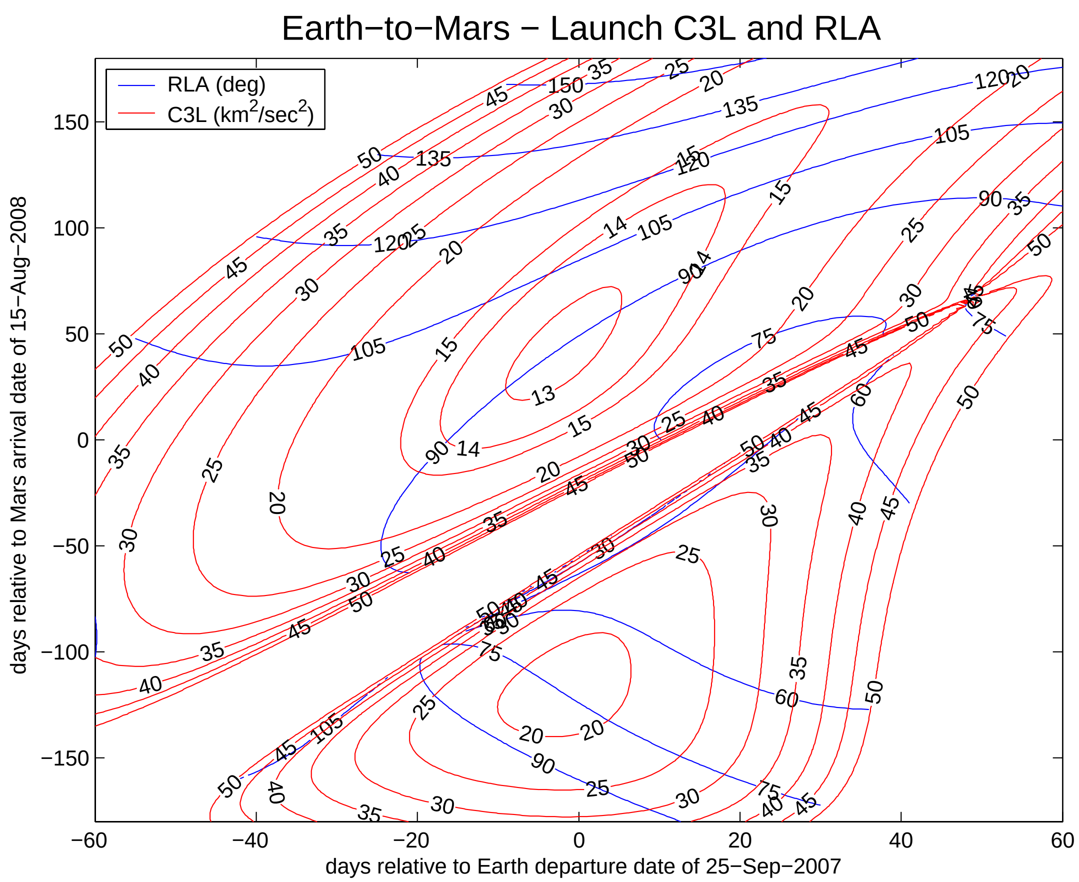
This next contour plot illustrates the behavior of the launch energy (C3L) and the interplanetary cruise flight time in days (TOF).



This next plot illustrates the behavior of the launch energy (C3L) and the declination of the departure hyperbola (DLA).



This final contour plot summarizes the behavior of the launch energy (C3L) and the right ascension of the launch hyperbola (RLA).



Technical Discussion

A solution for the launch and arrival impulsive delta-v vectors can be determined from the solution of the Lambert two-point boundary-value problem (TPBVP). Lambert's Theorem states that the time to traverse a trajectory depends only upon the length of the semimajor axis a of the transfer trajectory, the sum $r_i + r_f$ of the distances of the initial and final positions relative to a central body, and the length c of the chord joining these two positions.

Lambert's Problem

Lambert's problem is concerned with the determination of an orbit that passes between two positions within a specified time-of-flight. This classic astrodynamic problem is also known as the orbital two-point boundary value problem (TPBVP).

The time to traverse a trajectory depends only upon the length of the semimajor axis *a* of the transfer trajectory, the sum $r_i + r_f$ of the distances of the initial and final positions relative to a central body, and the length *c* of the chord joining these two positions. This relationship can be stated as follows:

$$tof = tof\left(r_i + r_f, c, a\right)$$

From the following form of Kepler's equation

$$t - t_0 = \sqrt{\frac{a^3}{\mu}}\left(E - e\sin E\right)$$

we can write

$$t = \sqrt{\frac{a^3}{\mu}}\left[E - E_0 - e\left(\sin E - \sin E_0\right)\right]$$

where *E* is the eccentric anomaly associated with radius *r*, E_0 is the eccentric anomaly at r_0, and $t = 0$ when $r = r_0$.

At this point we need to introduce the following trigonometric sun and difference identities:

$$\sin\alpha - \sin\beta = 2\sin\frac{\alpha - \beta}{2}\cos\frac{a + \beta}{2}$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha - \beta}{2}\sin\frac{a + \beta}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha - \beta}{2}\cos\frac{a + \beta}{2}$$

If we let $E = \alpha$ and $E_0 = \beta$ and substitute the first trig identity into the second equation above, we have the following equation:

$$t = \sqrt{\frac{a^3}{\mu}}\left\{E - E_0 - 2\sin\frac{E - E_0}{2}\left(e\cos\frac{E + E_0}{2}\right)\right\}$$

With the two substitutions given by

$$e\cos\frac{E + E_0}{2} = \cos\frac{\alpha + \beta}{2}$$

$$\sin\frac{E - E_0}{2} = \sin\frac{\alpha - \beta}{2}$$

the time equation becomes

$$t = \sqrt{\frac{a^3}{\mu}} \left\{ (\alpha - \beta) - 2\sin\frac{\alpha - \beta}{2} \cos\frac{\alpha + \beta}{2} \right\}$$

From the elliptic relationships given by

$$r = a(1 - e\cos E)$$

$$x = a(\cos E - e)$$

$$y = a\sin E\sqrt{1 - e^2}$$

and some more manipulation, we have the following equations:

$$\cos\alpha = \left(1 - \frac{r + r_0}{2a}\right) - \frac{c}{2a} = 1 - \frac{r + r_0 + c}{2a} = 1 - \frac{s}{a}$$

$$\sin\beta = \left(1 - \frac{r + r_0}{2a}\right) + \frac{c}{2a} = 1 - \frac{r + r_0 - c}{2a} = 1 - \frac{s - c}{a}$$

This part of the derivation makes use of the following three relationships:

$$\cos\frac{\alpha - \beta}{2} \cos\frac{\alpha + \beta}{2} = 1 - \frac{r + r_0}{2}$$

$$\sin\frac{\alpha - \beta}{2} \sin\frac{\alpha + \beta}{2} = \sin\frac{E - E_0}{2} \sqrt{1 - \left(e\cos\frac{E + E_0}{2}\right)^2}$$

$$\left(\sin\frac{\alpha - \beta}{2} \sin\frac{\alpha + \beta}{2}\right)^2 = \left(\frac{x - x_0}{2a}\right)^2 + \left(\frac{y - y_0}{2a}\right)^2 = \left(\frac{c}{2a}\right)^2$$

With the use of the half angle formulas given by

$$\sin\frac{\alpha}{2} = \sqrt{\frac{s}{2a}} \qquad \sin\frac{\beta}{2} = \sqrt{\frac{s - c}{2a}}$$

and several additional substitutions, we have the time-of-flight form of Lambert's theorem

$$t = \sqrt{\frac{a^3}{\mu}} \left[(\alpha - \beta) - (\sin\alpha - \sin\beta) \right]$$

A discussion about the angles α and β can be found in "Geometrical Interpretation of the Angles α and β in Lambert's Problem" by J. E. Prussing, AIAA *Journal of Guidance and Control*, Volume 2, Number 5, Sept.-Oct. 1979, pages 442-443.

The algorithm used in this MATLAB script is based on the method described in “A Procedure for the Solution of Lambert’s Orbital Boundary-Value Problem” by R. H. Gooding, *Celestial Mechanics and Dynamical Astronomy* **48:** 145-165, 1990. This iterative solution is valid for elliptic, parabolic and hyperbolic transfer orbits which may be either posigrade or retrograde, and involve one or more revolutions about the central body.

The $\Delta V's$ required at launch and arrival are simply the differences between the velocity on the transfer trajectory determined by the solution of Lambert’s problem and the heliocentric velocities of the two planets. If we treat each planet as a point mass and assume *impulsive* maneuvers, the *planet-centered* magnitude and direction of the required maneuvers are given by the two vector equations:

$$\Delta \mathbf{V}_L = \mathbf{V}_{T_L} - \mathbf{V}_{P_L}$$

$$\Delta \mathbf{V}_A = \mathbf{V}_{T_A} - \mathbf{V}_{P_A}$$

where

$\mathbf{V}_{T_L}$ = heliocentric velocity vector of the transfer trajectory at launch

$\mathbf{V}_{T_A}$ = heliocentric velocity vector of the transfer trajectory at arrival

$\mathbf{V}_{P_L}$ = heliocentric velocity vector of the launch planet

$\mathbf{V}_{P_A}$ = heliocentric velocity vector of the arrival planet

The scalar magnitude of each maneuver is also called the “hyperbolic excess velocity” or V_∞ at launch and arrival. The hyperbolic excess velocity is the speed of the spacecraft relative to each planet at an *infinite* distance from the planet. Furthermore, the *energy* or C_3 at launch or arrival is equal to V_∞^2 for the respective maneuver. C_3 is also equal to <u>twice</u> the orbital energy per unit mass (the specific orbital energy).

The orientation of the departure and arrival hyperbolas is specified in terms of the right ascension and declination of the asymptote. These coordinates can be calculated using the components of the V_∞ velocity vector.

The right ascension of the asymptote is determined from

$$\alpha = \tan^{-1}\left(\Delta V_y, \Delta V_z\right)$$

and the geocentric declination of the asymptote is given by

$$\delta = 90^0 - \cos^{-1}\left(\Delta \hat{V}_z\right)$$

where $\Delta \hat{V}_z$ is z-component of the unit ΔV vector.

In this script the heliocentric planetary coordinates and therefore the ΔV vectors are computed in the J2000 ecliptic and equinox coordinate system. In order to determine the orientation of the departure hyperbola, these ΔV vectors must be transformed to the equatorial frame.

The required transformation is given by

$$\Delta \mathbf{V}_{eq} = \begin{bmatrix} 1 & -0.000000479966 & 0 \\ 0.000000440360 & 0.917482137087 & 0.397776982902 \\ -0.000000190919 & -0.397776982902 & 0.917482137087 \end{bmatrix} \Delta \mathbf{V}_{ec}$$

where $\Delta \mathbf{V}_{ec}$ is the delta-velocity vector in the ecliptic frame, and $\Delta \mathbf{V}_{eq}$ is the delta-velocity vector in the equatorial frame.

This MATLAB script models the planetary coordinates using the algorithm described in chapter 30 of *Astronomical Algorithms* by Jean Meeus. Each orbital element is represented by a cubic polynomial of the form

$$a_0 + a_1 T + a_2 T^2 + a_3 T^3$$

where the fundamental time argument T is given by

$$T = \frac{JD - 2451545}{36525}$$

In this expression JD is the Julian date.

References and Bibliography

"Modern Astrodynamics", Victor R. Bond and Mark C. Allman, Princeton University Press, 1996.

"Interplanetary Mission Design Handbook, Volume 1, Part 2", JPL Publication 82-43, September 15, 1983.

Explanatory Supplement to the Astronomical Almanac, Edited by P. K. Seidelmann, University Science Books, 1992.

"Update to Mars Coordinate Frame Definitions", R. A. Mase, JPL IOM 312.B/015-99, 15 July 1999.

R. H. Battin, *An Introduction to the Mathematics and Methods of Astrodynamics*, AIAA, 1987.