A Matlab Script for Parametric Analysis of Minimum TLI Delta-V, Two-Body Lunar Transfer Trajectories

This document is the user's guide for a Matlab script called `tli_sweep` that can be used to perform a parametric analysis of two-body lunar transfer trajectories. The software assumes that trans-lunar injection (TLI) occurs *impulsively* from a circular Earth park orbit. The software solves for the minimum TLI delta-v using a two-body Lambert solution for the transfer trajectory from the Earth park orbit to the center of the moon.

This computer program uses the SNOPT nonlinear programming (NLP) method to solve this classic trajectory optimization problem. The lunar coordinates required by the software are computed using the JPL DE421 ephemeris.

Input data file

The `tli_sweep` Matlab script is "data-driven" by a simple text file created by the user. This section describes a typical input data file. In the following discussion the actual input file contents are in `courier` font and all explanations are in *times italic* font.

Each data item within an input file is preceded by one or more lines of *annotation* text. Do not delete any of these annotation lines or increase or decrease the number of lines reserved for each comment. However, you may change them to reflect your own explanation. The annotation line also includes the correct units and when appropriate, the valid range of the input.

The first four lines of any input file are reserved for user comments. These lines are ignored by the software. However the input file must begin with four and only four initial text lines.

```
*****************************************
* data file for tli_sweep.m Matlab script
* tli_sweep1.in    March 22, 2007
*****************************************
```

The first inputs define the calendar date of the TLI maneuver. Be sure to include all four digits of the calendar year.

```
initial calendar date (month, day, year)
1, 1, 2008
```

The next input specifies the type of TLI maneuver. Please see the Technical Discussion later in this document for an explanation of this maneuver.

```
type of TLI maneuver (1 = ascending, 2 = descending)
2
```

The next two inputs define the value of the altitude and orbital inclination of the circular park orbit.

```
park orbit altitude (kilometers)
185.2

park orbit inclination (degrees)
28.5
```

The duration of the lunar transfer trajectory is set by this next input.

```
transfer time (hours)
84.0
```

The total simulation duration and time step size of the parametric sweep are specified using these next two inputs.

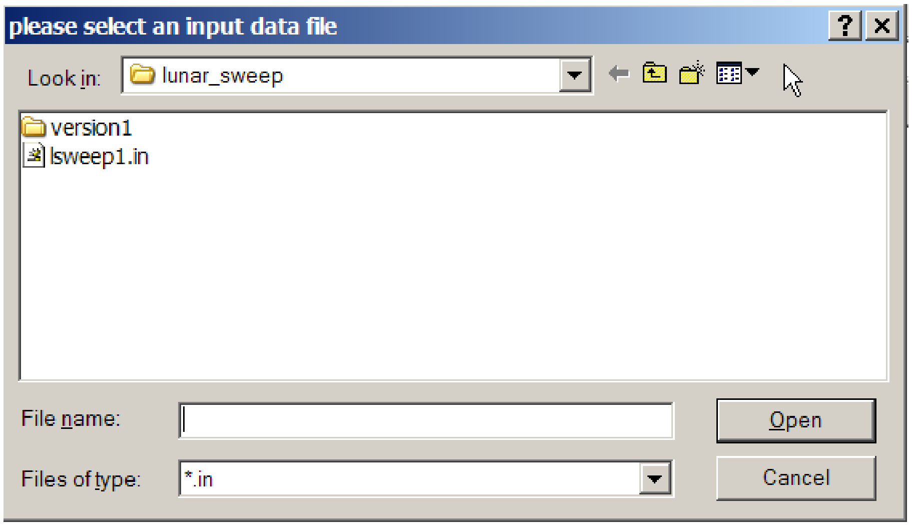
```
simulation duration (days)
90

simulation step size (days)
0.25
```

The final input is the name of the data file created by the software.

```
name of summary data file
tli_sweep1.txt
```

Running the script

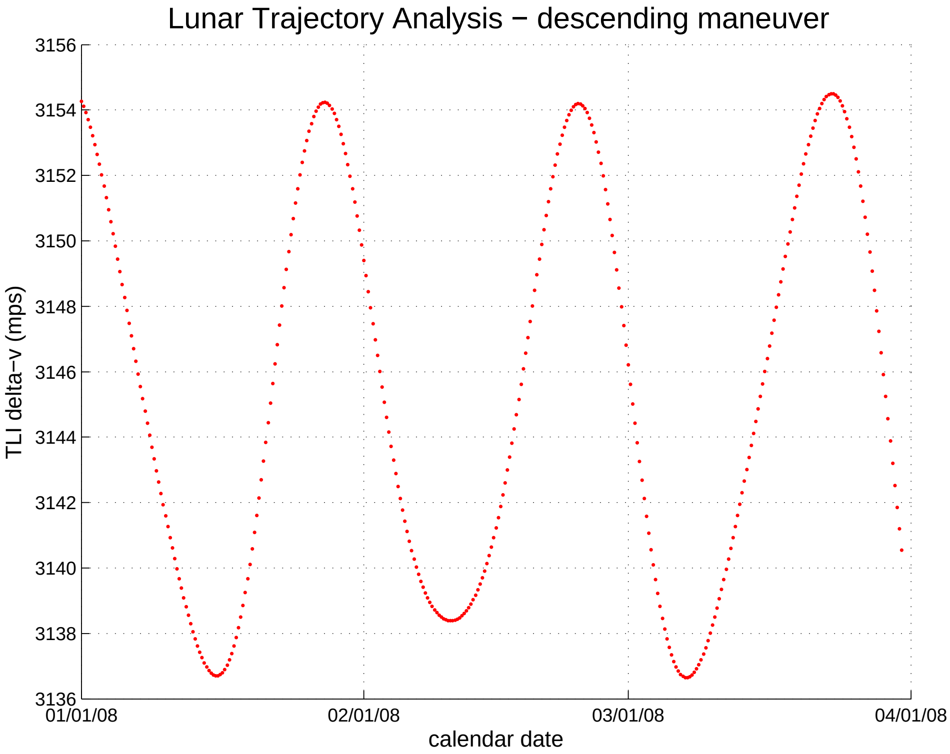
When the `tli_sweep` script is started, the software will display the following screen which allows the user to select a data file for processing.



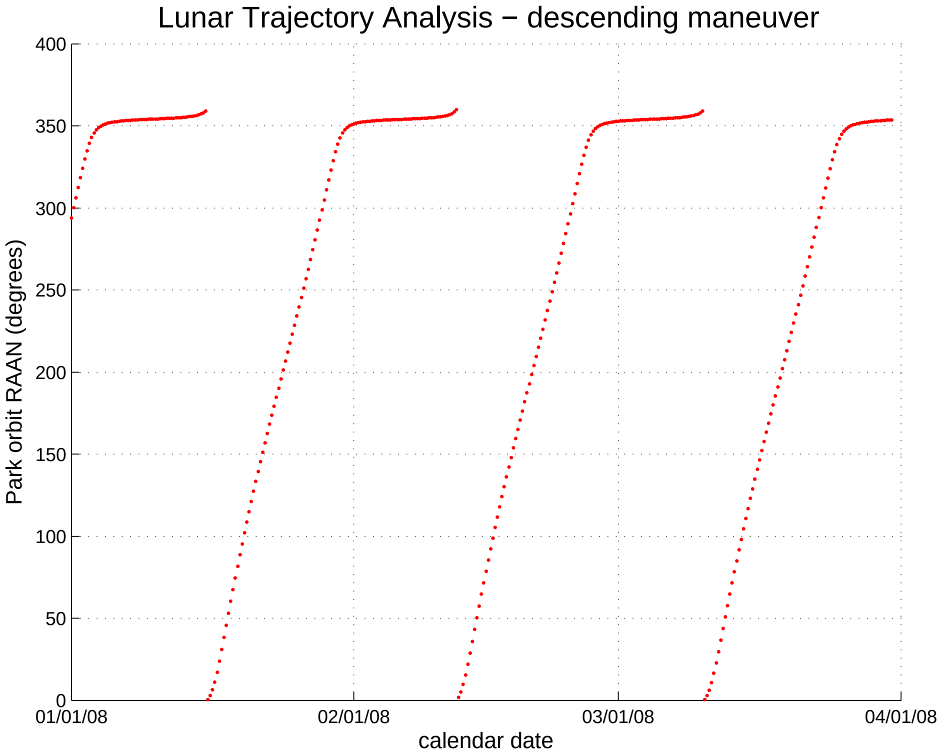
The file type defaults to names with a `*.in` filename extension. However, you can select any `tli_sweep` compatible ASCII data file.

Program example

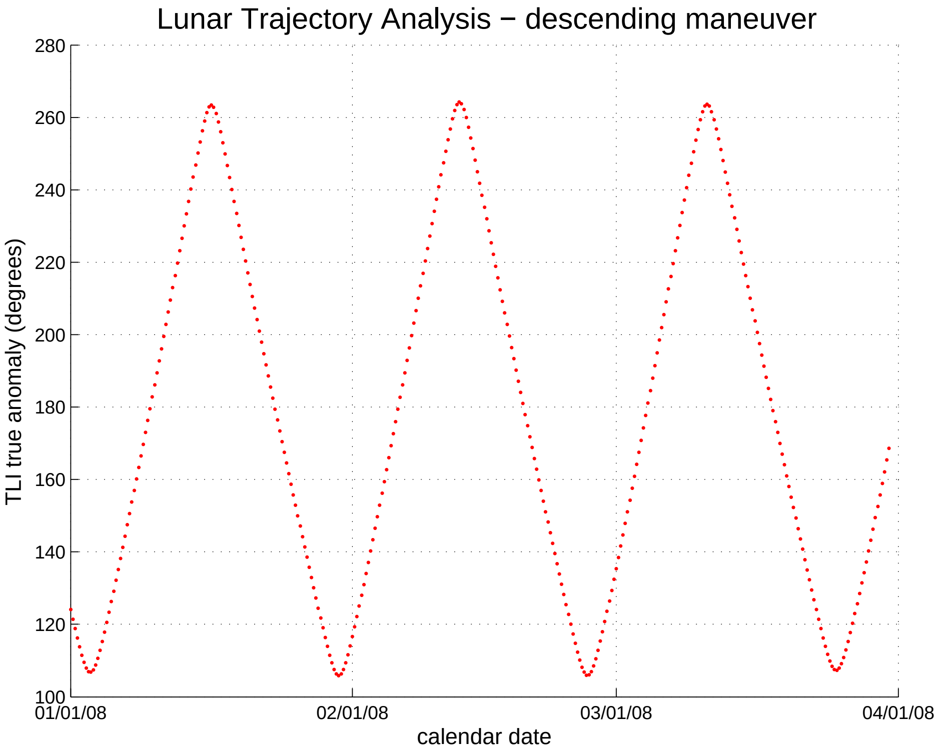
This section contains plots of the behavior of the TLI characteristics for a descending node transfer for a period of three months at a step size of 0.25 days. The initial calendar date and time is 0 hours TDB on January 1, 2008 and the transfer time from TLI until lunar encounter is 84 hours. The first plot shows the variation of the magnitude of the TLI impulsive delta-v as a function of the calendar date.



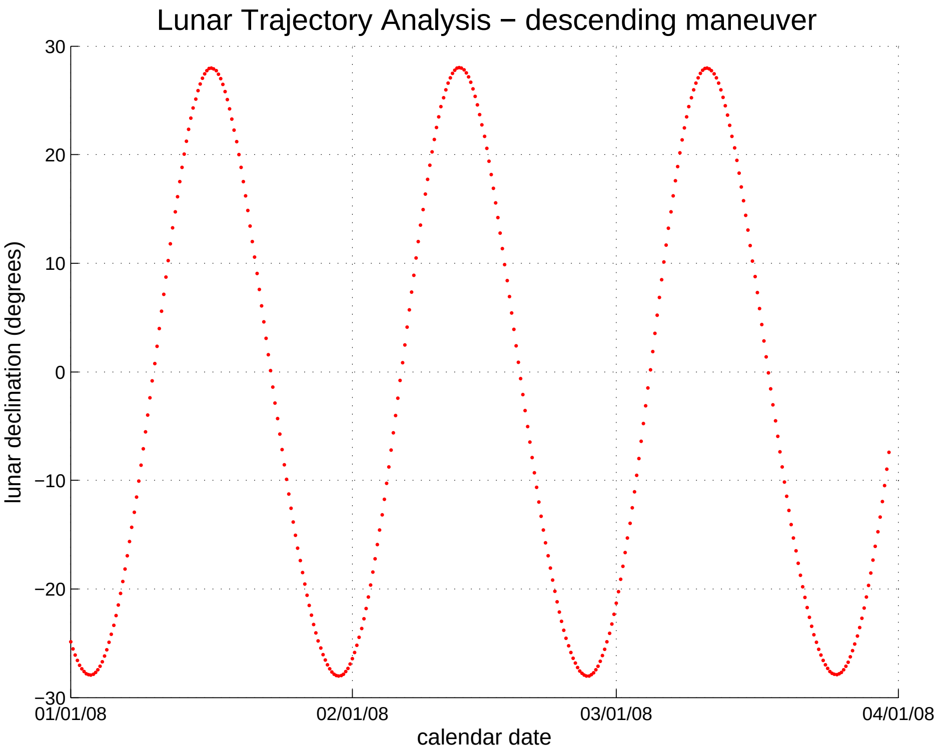
This next plot illustrates the variation of the RAAN of the circular park orbit as a function of the TLI calendar date.



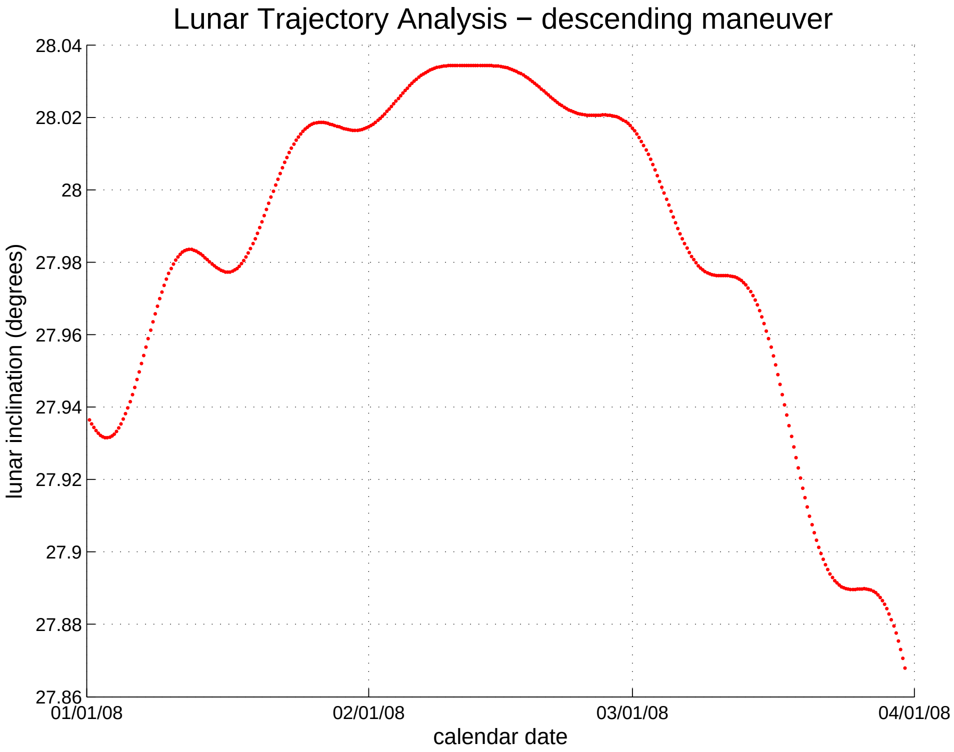
This plot illustrates the behavior of the true anomaly of the impulsive TLI maneuver on the circular park orbit.



This next plot shows the geocentric declination of the moon at the encounter time.



This final plot illustrates the evolution of the geocentric orbital inclination of the moon.



The following printout illustrates the first eight days of the ASCII data file created by the software.

```
time (days)    delta-v (mps)      RAAN (deg)       tanom (deg)      C3 (km/sec)^2   rasc moon (deg)   decl moon (deg)
     0.0000    3154.26737087     293.97720168     124.15265786       -1.61934856     235.30494550      -24.88171236
     0.2500    3154.10755774     300.07998669     121.43863512       -1.62284758     238.51328259      -25.52261158
     0.5000    3153.92108565     306.18899458     118.78003988       -1.62693022     241.76185305      -26.09405035
     0.7500    3153.70956707     312.27105277     116.20237412       -1.63156114     245.04889036      -26.59310552
     1.0000    3153.47466323     318.27587296     113.74525870       -1.63670395     248.37203799      -27.01699195
     1.2500    3153.21807014     324.12620286     111.47115679       -1.64232148     251.72836258      -27.36310665
     1.5000    3152.94150459     329.70348264     109.47812284       -1.64837611     255.11438384      -27.62907362
     1.7500    3152.64669058     334.83384588     107.91237414       -1.65483008     258.52612185      -27.81278819
     2.0000    3152.33534590     339.29720864     106.96076958       -1.66164574     261.95916197      -27.91245900
     2.2500    3152.00916934     342.90037697     106.78724986       -1.66878588     265.40873611      -27.92664613
     2.5000    3151.66982838     345.59681652     107.42827767       -1.67621397     268.86981815      -27.85429366
     2.7500    3151.31894760     347.51770675     108.76580390       -1.68389442     272.33723019      -27.69475548
     3.0000    3150.95809788     348.87041621     110.61617181       -1.69179282     275.80575572      -27.44781301
     3.2500    3150.58878652     349.83824198     112.81769985       -1.69987616     279.27025487      -27.11368444
     3.5000    3150.21244835     350.55047096     115.25658720       -1.70811303     282.72577716      -26.69302501
     3.7500    3149.83043793     351.09098660     117.85912117       -1.71647375     286.16766702      -26.18691865
     4.0000    3149.44402299     351.51330030     120.57835280       -1.72493057     289.59165809      -25.59686166
     4.2500    3149.05437905     351.85197000     123.38400735       -1.73345776     292.99395329      -24.92473926
     4.5000    3148.66258545     352.12984663     126.25608979       -1.74203168     296.37128831      -24.17279654
     4.7500    3148.26962268     352.36245136     129.18102625       -1.75063089     299.72097775      -23.34360505
     5.0000    3147.87637123     352.56061151     132.14934287       -1.75923610     303.04094375      -22.44002675
     5.2500    3147.48361174     352.73207185     135.15424927       -1.76783024     306.32972830      -21.46517665
     5.5000    3147.09202675     352.88250179     138.19075473       -1.77639837     309.58649079      -20.42238568
     5.7500    3146.70220373     353.01614031     141.25510436       -1.78492764     312.81099308      -19.31516495
     6.0000    3146.31463962     353.13621913     144.34441082       -1.79340719     316.00357459      -18.14717236
     6.2500    3145.92974670     353.24524687     147.45640856       -1.80182799     319.16511990      -16.92218251
     6.5000    3145.54785963     353.34520404     150.58928667       -1.81018275     322.29702133      -15.64406028
     6.7500    3145.16924384     353.43767974     153.74157300       -1.81846564     325.40113864      -14.31673867
     7.0000    3144.79410489     353.52396921     156.91205262       -1.82667219     328.47975778      -12.94420090
     7.2500    3144.42259882     353.60514493     160.09970917       -1.83479899     331.53555002      -11.53046696
     7.5000    3144.05484329     353.68210933     163.30368231       -1.84284348     334.57153281      -10.07958448
     7.7500    3143.69092941     353.75563458     166.52323564       -1.85080366     337.59103280       -8.59562377
     8.0000    3143.33093406     353.82639333     169.75773259       -1.85867787     340.59765177       -7.08267690
```

A guide to the items captured in this data file is as follows;

time (days) = elapsed time from initial calendar date in days

delta-v (mps) = magnitude of impulsive TLI maneuver in meters per second

RAAN (deg) = park orbit right ascension of the ascending node in degrees

tanom (deg) = park orbit true anomaly of the TLI maneuver in degrees

C3 (km/sec)^2 = twice the specific orbital energy in kilometers^2/second^2

rasc moon (deg) = geocentric right ascension of the moon in degrees at arrival

decl moon (deg) = geocentric declination of the moon in degrees at arrival

Please note that all angular coordinates are measured with respect to the Earth mean equator and equinox of J2000 (EME2000) coordinate system.

Technical Discussion

This section describes several of the major algorithms implemented in the `tli_sweep` Matlab script.

Nonlinear programming problem

A trajectory optimization problem can be described by a system of *dynamic variables*

$$\mathbf{z} = \begin{bmatrix} \mathbf{y}(t) \\ \mathbf{u}(t) \end{bmatrix}$$

consisting of the *state variables* $\mathbf{y}$ and the *control variables* $\mathbf{u}$ for any time t. In this discussion vectors are denoted in bold.

The system dynamics are defined by a vector system of ordinary differential equations called the *state equations* that can be represented as follows:

$$\dot{\mathbf{y}} = \frac{d\mathbf{y}}{dt} = \mathbf{f}\left[\mathbf{y}(t), \mathbf{u}(t), \mathbf{p}, t\right]$$

where $\mathbf{p}$ is a vector of problem *parameters* that is not time dependent.

The initial dynamic variables at time t_0 are defined by $\boldsymbol{\psi}_0 \equiv \boldsymbol{\psi}\left[\mathbf{y}(t_0), \mathbf{u}(t_0), t_0\right]$ and the terminal conditions at the final time t_f are defined by $\boldsymbol{\psi}_f \equiv \boldsymbol{\psi}\left[\mathbf{y}(t_f), \mathbf{u}(t_f), t_f\right]$. These conditions are called the *boundary values* of the trajectory problem. The problem may also be subject to *path constraints* of the form $\mathbf{g}\left[\mathbf{y}(t), \mathbf{u}(t), t\right] = 0$.

The basic nonlinear programming problem (NLP) is to determine the control vector history and problem parameters that minimize the scalar performance index or objective function given by

$$J = \phi\left[\mathbf{y}(t_0), t_0, \mathbf{y}(t_f), t_f, \mathbf{p}\right]$$

while satisfying all the user-defined mission constraints.

During the two-body trajectory optimization, the main control variable is the park orbit true anomaly at the time of the TLI maneuver. The objective function or performance index is the scalar magnitude of the TLI delta-v vector. The final boundary conditions are the components of the moon's inertial position vector at encounter.

Park orbit true anomaly

An initial guess for the park orbit true anomaly at the time of the impulsive TLI maneuver is obtained iteratively. This iteration involves setting the park orbit true anomaly equal to the argument of perigee of the TLI maneuver of the Earth-to-Moon elliptical transfer orbit. This process is repeated until the change in true anomaly between successive iterations is small. The argument of perigee of the transfer orbit is determined by resolving the two-body Lambert problem which is a function of the current park orbit position vector and the position vector of the Moon at encounter.

For the first calendar date of the parametric sweep, the initial true anomaly guess is set to zero. Subsequent days in the sweep use the previous optimal true anomaly for their initial guess.

During the numerical optimization, the true anomaly is bounded according to

$$\theta_1 - 10^\circ \le \theta \le \theta_1 + 10^\circ$$

where θ_1 is the true anomaly computed from the initial guess iteration.

Park orbit RAAN

For a given TLI calendar date, there are two possible locations on the initial park orbit at which to perform the propulsive maneuver. One opportunity occurs during the ascending part of the park orbit and the other during the descending motion. The park orbit RAAN Ω_p at these two locations can be determined from spherical trigonometry relationships involving the park orbit inclination and the geocentric right ascension and declination of the moon at encounter.

In this Matlab script, the park orbit RAAN is held fixed during the numerical optimization. The RAAN option used is selected by the user.

The equations implemented in this Matlab script are as follows:

ascending maneuver

$$\Omega_p = -180^\circ + \alpha_m + \sin^{-1}\left(\frac{\tan\delta_m}{\tan i_p}\right)$$

descending maneuver

$$\Omega_p = \alpha_m - \sin^{-1}\left(\frac{\tan\delta_m}{\tan i_p}\right)$$

where

$\alpha_m =$ right ascension of the moon at encounter
$\delta_m =$ declination of the moon at encounter
$i_p =$ park orbit inclination

Solving the two body Lambert problem

Lambert's problem is concerned with the determination of an orbit that passes between two positions within a specified time-of-flight. This classic astrodynamic problem is also known as the orbital two-point boundary value problem (TPBVP).

The time to traverse a trajectory depends only upon the length of the semimajor axis *a* of the transfer trajectory, the sum $r_i + r_f$ of the distances of the initial and final positions relative to a central body, and the length *c* of the chord joining these two positions. This relationship can be stated as follows:

$$tof = tof\left(r_i + r_f, c, a\right)$$

From the following form of Kepler's equation

$$t - t_0 = \sqrt{\frac{a^3}{\mu}}\left(E - e\sin E\right)$$

we can write

$$t = \sqrt{\frac{a^3}{\mu}}\left[E - E_0 - e\left(\sin E - \sin E_0\right)\right]$$

where *E* is the eccentric anomaly associated with radius *r*, E_0 is the eccentric anomaly at r_0, and $t = 0$ when $r = r_0$.

At this point we need to introduce the following trigonometric sun and difference identities:

$$\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{a+\beta}{2}$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha-\beta}{2}\sin\frac{a+\beta}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha-\beta}{2}\cos\frac{a+\beta}{2}$$

If we let $E = \alpha$ and $E_0 = \beta$ and substitute the first trig identity into the second equation above, we have the following equation:

$$t = \sqrt{\frac{a^3}{\mu}}\left\{E - E_0 - 2\sin\frac{E-E_0}{2}\left(e\cos\frac{E+E_0}{2}\right)\right\}$$

With the two substitutions given by

$$e\cos\frac{E+E_0}{2}=\cos\frac{\alpha+\beta}{2}$$

$$\sin\frac{E-E_0}{2}=\sin\frac{\alpha-\beta}{2}$$

the time equation becomes

$$t=\sqrt{\frac{a^3}{\mu}}\left\{(\alpha-\beta)-2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}\right\}$$

From the elliptic relationships given by

$$r=a(1-e\cos E)$$

$$x=a(\cos E-e)$$

$$y=a\sin E\sqrt{1-e^2}$$

and some more manipulation, we have the following equations:

$$\cos\alpha=\left(1-\frac{r+r_0}{2a}\right)-\frac{c}{2a}=1-\frac{r+r_0+c}{2a}=1-\frac{s}{a}$$

$$\sin\beta=\left(1-\frac{r+r_0}{2a}\right)+\frac{c}{2a}=1-\frac{r+r_0-c}{2a}=1-\frac{s-c}{a}$$

This part of the derivation makes use of the following three relationships:

$$\cos\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}=1-\frac{r+r_0}{2}$$

$$\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2}=\sin\frac{E-E_0}{2}\sqrt{1-\left(e\cos\frac{E+E_0}{2}\right)^2}$$

$$\left(\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2}\right)^2=\left(\frac{x-x_0}{2a}\right)^2+\left(\frac{y-y_0}{2a}\right)^2=\left(\frac{c}{2a}\right)^2$$

With the use of the half angle formulas given by

$$\sin\frac{\alpha}{2}=\sqrt{\frac{s}{2a}}\qquad\qquad\sin\frac{\beta}{2}=\sqrt{\frac{s-c}{2a}}$$

and several additional substitutions, we have the time-of-flight form of Lambert’s theorem

$$t = \sqrt{\frac{a^3}{\mu}}\left[(\alpha - \beta) - (\sin\alpha - \sin\beta)\right]$$

A discussion about the angles α and β can be found in “Geometrical Interpretation of the Angles α and β in Lambert’s Problem” by J. E. Prussing, AIAA *Journal of Guidance and Control,* Volume 2, Number 5, Sept.-Oct. 1979, pages 442-443.

The algorithm used in this MATLAB script is based on the method described in “A Procedure for the Solution of Lambert’s Orbital Boundary-Value Problem” by R. H. Gooding, *Celestial Mechanics and Dynamical Astronomy* **48:** 145-165, 1990. This iterative solution is valid for elliptic, parabolic and hyperbolic transfer orbits which may be either posigrade or retrograde, and involve one or more revolutions about the central body.

References and Bibliography

“Lunar Trajectories”, NASA TN D-866, August 1961.

“Earth-Moon Trajectories”, JPL Technical Report No. 32-503, May 1, 1964.

“Three-Dimensional Lunar Trajectories”, V. A. Egorov, Mechanics of Space Flight Series, Israel Program for Scientific Translations, Jerusalem 1969.

“Circumlunar Trajectory Calculations”, MIT Instrumentation Laboratory Report R-353, April 1962.

“Optimal Low Thrust Trajectories to the Moon”, John T. Betts and Sven O. Erb, *SIAM Journal on Applied Dynamical Systems,* Vol. 2, No. 2, pp. 144-170, 2003.

“Integrated Algorithm for Lunar Transfer Trajectories Using a Pseudostate Technique”, R. V. Ramanan, AIAA *Journal of Guidance, Control and Dynamics,* Vol. 25, No. 5, September-October 2002, pp. 946-952.

“Nonimpact Lunar Transfer Trajectories Using the Pseudostate Technique”, R. V. Ramanan and V. Adimurthy, AIAA *Journal of Guidance, Control and Dynamics*, Vol. 28, No. 2, March-April 2005, pp. 217-225.

“Injection Conditions for Lunar Trajectories”, R. Kolenkiewicz and W. Putney, NASA TM X-55390, November 1965.

“Coplanar Three-Body Trans-Earth Lunar Trajectory Simulation Methodology”, H. Ikawa, AIAA 88-0381, AIAA 26th Aerospace Sciences Meeting, Reno, Nevada, January 11-14, 1988.

“Earth-Moon Trajectories, 1964-69”, R. J. Richard, V. C. Clarke, Jr., R. Y. Roth and W. E. Kirhofer, JPL Technical Report No. 32-503, May 1, 1964.

“User’s Guide for SNOPT Version 6, A Fortran Package for Large-Scale Nonlinear Programming”, Philip E. Gill, Walter Murray and Michael A. Saunders, December 2002.