Computer Methods for TransEarth TCM
Delta-V Design and Optimization

This document describes three computer programs that can be used to design and optimize an impulsive trajectory correction maneuver (TCM) during the transEarth phase of lunar flight. The first algorithm “targets” the maneuver to achieve Earth relative flight path coordinates at the Earth entry interface (EI). The other two numerical methods attempt to minimize the magnitude of the TCM delta-v and also achieve the EI targets or mission constraints.

The following is a summary of the numerical methods used in each computer program.

1. System of nonlinear constraint equations with a simple shooting method
2. Nonlinear optimization with a simple shooting method
3. Nonlinear optimization with collocation and direct transcription method

Each computer program is data-driven by a simple ASCII input file created by the user. The software allows the user to specify the order and degree of the Earth gravity model used in the geocentric equations of motion. The user also has the option to include point-mass gravity perturbations of the Sun and Moon. The solar and lunar ephemeris is based on the JPL DE421 development ephemeris.

The user can choose a subset of EI flight path coordinates that includes the geodetic altitude, geodetic latitude, geographic longitude, and the relative azimuth angle. Please note that the relative flight path angle must be specified for all three algorithms. Finally, the user can also provide an initial guess for the three components of the TCM delta-v maneuver.

INPUT DATA FILE

This section illustrates a typical input data file for the TCM software suite.

```
*****************************************
** transearth TCM trajectory optimization
** n-body geocentric motion
** Moon-to-Earth data file - tcm1.in
** July 22, 2008
*****************************************

TCM epoch
Aug 6 2018 15:59:59.994 TDB

************************************************
geocentric EME2000 orbital elements prior to TCM
************************************************

semimajor axis (kilometers)
0.220615448822D+06

orbital eccentricity (non-dimensional)
0.970867462750D+00

orbital inclination (degrees)
0.509115579289D+02
```

argument of perigee (degrees)
0.347338533437D+03

right ascension of the ascending node (degrees)
0.212814404333D+03

true anomaly (degrees)
0.198500745260D+03

****
initial guess and bounds for geocentric TCM delta-v vector
****

x-component of TCM velocity vector (meters/second)
0.0

y-component of TCM velocity vector (meters/second)
0.0

z-component of TCM velocity vector (meters/second)
0.0

****
entry interface constraints (set to 1.0d99 to ignore)
****

geodetic altitude (kilometers)
121.92

relative flight path angle (degrees)
-6.2

geodetic latitude (degrees)
-19.5

east longitude (degrees)
121.0

relative azimuth (degrees)
1.0d99

****
trajectory perturbations
****

name of Earth gravity model data file
egm96.dat

order of Earth gravity model (zonals)
8

degree of Earth gravity model (tesserals)
8

include solar perturbations (1 = yes, 0 = no)
1

include lunar perturbations (1 = yes, 0 = no)
1

****
root-finding and integration algorithm control
****

```
root-finding tolerance
1.0d-8

RKF7(8) truncation error tolerance
1.0d-12

nonlinear equations tolerance
1.0d-8
```

PROGRAM OUTPUT

The following is a summary of the program output from each computer simulation for this example. The entry interface targets for this example were the geodetic altitude (121.92 kilometers) and geodetic latitude (-19.5 degrees), and the geographic east longitude (121 degrees). The EI relative flight path angle target is -6.2 degrees.

System of nonlinear equations

The following is the delta-v summary and the targets achieved at the entry interface.

```
======================
program transearth_tcm
======================

----------------------------------
time and conditions prior to TCM
(geocentric - EME2000 coordinates)
----------------------------------

UTC epoch            2018 AUG 06 15:58:54.81085

TDB Julian date      2458337.166666597127914


     sma (km)             eccentricity        inclination (deg)       argper (deg)
 0.220615448822D+06    0.970867462750D+00    0.509115579289D+02    0.347338533437D+03

    raan (deg)         true anomaly (deg)       arglat (deg)          period (min)
 0.212814404333D+03    0.198500745260D+03    0.185839278697D+03    0.171875356468D+05

     rx (km)                ry (km)               rz (km)              rmag (km)
 0.127984235359D+06    0.947167455303D+05    -.126124998721D+05    0.159719446334D+06

     vx (kps)              vy (kps)              vz (kps)             vmag (kps)
 -.157405686429D+01    -.814842228516D+00    -.207047943056D+00    0.178451442138D+01

----------------------------------
time and conditions after the TCM
(geocentric - EME2000 coordinates)
----------------------------------

UTC epoch            2018 AUG 06 15:58:54.81085

TDB Julian date      2458337.166666597127914

     sma (km)             eccentricity        inclination (deg)       argper (deg)
 0.218504581777D+06    0.970593697062D+00    0.520348622425D+02    0.347302049819D+03

    raan (deg)         true anomaly (deg)       arglat (deg)          period (min)
 0.212960097719D+03    0.198446474919D+03    0.185748524738D+03    0.169414489839D+05
```

rx (km)	ry (km)	rz (km)	rmag (km)
0.127984235359D+06	0.947167455303D+05	-.126124998721D+05	0.159719446334D+06

vx (kps)	vy (kps)	vz (kps)	vmag (kps)
-.156626163169D+01	-.817656773616D+00	-.212836396865D+00	0.177961721450D+01

TCM delta-v vector and magnitude
(geocentric EME2000 coordinates)

delta-vx	7.795232601052531 meters/second
delta-vy	-2.814545100198722 meters/second
delta-vz	-5.788453808476387 meters/second
deltav	10.109080715793580 meters/second
pitch angle	29.868180335298231 degrees
yaw angle	269.983507056383473 degrees

time and conditions at entry interface
(geocentric - EME2000 coordinates)

UTC epoch	2018 AUG 07 08:00:24.72001
TDB Julian date	2458337.834373878780752

sma (km)	eccentricity	inclination (deg)	argper (deg)
0.223705147069D+06	0.971280418493D+00	0.519434659012D+02	0.347329965384D+03

raan (deg)	true anomaly (deg)	arglat (deg)	period (min)
0.212885279070D+03	0.347743436481D+03	0.335073401866D+03	0.175498615683D+05

rx (km)	ry (km)	rz (km)	rmag (km)
-.586479273288D+04	-.178173078828D+04	-.215629990858D+04	0.649769095120D+04

vx (kps)	vy (kps)	vz (kps)	vmag (kps)
0.492973713149D+00	-.731828662427D+01	0.819193017339D+01	0.109958202133D+02

flight path coordinates at entry interface
(Earth relative geocentric - EME2000)

geodetic altitude	121.920000000021901 kilometers
geodetic latitude	-19.500000000000188 degrees
east longitude	120.999999999999829 degrees
flight path angle	-6.200000000001041 degrees
relative azimuth	38.982987221369214 degrees
relative velocity	10.710749566513554 km/sec

orbital elements and state vector at entry interface
(geocentric - EME2000 coordinates)

```
UTC epoch            2018 AUG 07 08:00:24.72001

TDB Julian date      2458337.834373878780752

     sma (km)            eccentricity       inclination (deg)       argper (deg)
 0.223705147069D+06    0.971280418493D+00    0.519434659012D+02    0.347329965384D+03

    raan (deg)         true anomaly (deg)       arglat (deg)          period (min)
 0.212885279070D+03    0.347743436481D+03    0.335073401866D+03    0.175498615683D+05

     rx (km)               ry (km)               rz (km)              rmag (km)
 -.586479273288D+04    -.178173078828D+04    -.215629990858D+04    0.649769095120D+04

     vx (kps)              vy (kps)              vz (kps)             vmag (kps)
 0.492973713149D+00    -.731828662427D+01    0.819193017339D+01    0.109958202133D+02
```

Optimization with shooting method

The following is the delta-v summary and the targets achieved at the entry interface using this numerical method.

```
--------------------------------
TCM delta-v vector and magnitude
(geocentric EME2000)
--------------------------------

delta-vx                  7.795236204346402 meters/second
delta-vy                 -2.814543163808415 meters/second
delta-vz                 -5.788454575706034 meters/second

deltav                   10.109083394527207 meters/second

pitch angle              29.868198388869313 degrees

yaw angle               269.983507074416423 degrees

flight path coordinates
-----------------------

geodetic altitude       121.920000001642620 kilometers

geodetic latitude       -19.500000000001112 degrees

east longitude          120.999999729731172 degrees

flight path angle        -6.200000000001102 degrees

relative azimuth         38.982987117364509 degrees

relative velocity        10.710749566463926 km/sec
```

Optimization with collocation method

The following is the delta-v summary and the targets achieved at the entry interface.

```
TCM delta-v vector and magnitude

delta-vx                  7.795234075235853 meters/second
delta-vy                 -2.814552459976289 meters/second
delta-vz                 -5.788463341273181 meters/second
```

```
deltav                  10.109089360122415 meters/second

pitch angle             29.868136313642808 degrees

yaw angle              269.983531314061622 degrees

flight path coordinates
-----------------------

geodetic altitude      121.920000000216532 kilometers

geodetic latitude      -19.500000000000028 degrees

east longitude         121.000000542124639 degrees

flight path angle       -6.200000000009567 degrees

relative azimuth        38.982985574963863 degrees

relative velocity       10.710749577250612 km/sec
```

TECHNICAL DISCUSSION

This section describes the fundamental algorithms implemented in this software suite. It includes a summary of the simple shooting method used in the first two computer programs, the equations of motion, and the transformation of the inertial state vector at the entry interface to the corresponding flight path coordinates which are targets or mission constraints. In these applications, the fundamental time system is barycentric dynamical time (TDB) and the coordinate system is the Earth mean equator and equinox of J2000 (EME2000).

The first program attempts to solve a system of three nonlinear constraint equations similar to

$$h_p - h_t = 0 \qquad \phi_p - \phi_t = 0 \qquad \lambda_p - \lambda_t = 0$$

where the *p* subscript denotes a predicted coordinate and the *t* subscript is a user-defined target.

The other two programs attempt to minimize a scalar performance index given by

$$J = \left|\Delta \bar{V}\right| = \sqrt{\Delta V_x^2 + \Delta V_y^2 + \Delta V_z^2}$$

where $\Delta V_x, \Delta V_y, \Delta V_z$ are the components of the impulsive TCM delta-v.

Shooting method

The elapsed time from the TCM until the spacecraft reaches the entry interface with the user-defined relative flight path angle is determined by an algorithm that includes a Runge-Kutta-Fehlberg 7(8) numerical integration method embedded within Brent’s one-dimensional root-finder. This technique searches for the time at which the difference between the predicted and user-defined relative flight path angles is within a small tolerance $\left(\gamma_p - \gamma_t < \varepsilon\right)$.

Geocentric equations of motion

The TCM computer programs implement a *special perturbation* technique which numerically integrates the vector system of second-order, nonlinear differential equations of motion of a spacecraft given by

$$\mathbf{a}(\mathbf{r},t)=\ddot{\mathbf{r}}(\mathbf{r},t)=\mathbf{a}_g(\mathbf{r},t)+\mathbf{a}_m(\mathbf{r},t)+\mathbf{a}_s(\mathbf{r},t)$$

where

$t =$ barycentric dynamical time

$\mathbf{r} =$ inertial position vector of the spacecraft

$\mathbf{a}_g =$ acceleration due to the Earth's gravity

$\mathbf{a}_m =$ acceleration due to the Moon

$\mathbf{a}_s =$ acceleration due to the Sun

This computer program uses a *spherical harmonic* representation of the Earth's geopotential function given by

$$\Phi(r,\phi,\lambda)=\frac{\mu}{r}+\frac{\mu}{r}\sum_{n=1}^{\infty}C_n^0\left(\frac{R}{r}\right)^n P_n^0(u)+\frac{\mu}{r}\sum_{n=1}^{\infty}\sum_{m=1}^{n}\left(\frac{R}{r}\right)^n P_n^m(u)\left[S_n^m\sin m\lambda+C_n^m\cos m\lambda\right]$$

where ϕ is the geocentric latitude of the spacecraft, λ is the geocentric east longitude of the spacecraft and $r=|\mathbf{r}|=\sqrt{x^2+y^2+z^2}$ is the geocentric distance of the spacecraft. In this expression the *S's* and *C's* are *unnormalized* harmonic coefficients of the geopotential, and the *P's* are associated Legendre polynomials of degree *n* and order *m* with argument $u=\sin\phi$.

The software calculates the spacecraft's acceleration due to the Earth's gravity field with a vector equation derived from the gradient of the potential function expressed as $\mathbf{a}_g(\mathbf{r},t)=\nabla\Phi(\mathbf{r},t)$.

This acceleration vector is a combination of pure two-body or *point mass* gravity acceleration and the gravitational acceleration due to higher order nonspherical terms in the Earth's geopotential. In terms of the Earth's geopotential Φ, the inertial rectangular cartesian components of the spacecraft's acceleration vector are as follows:

$$\ddot{x}=\left(\frac{1}{r}\frac{\partial\Phi}{\partial r}-\frac{z}{r^2\sqrt{x^2+y^2}}\frac{\partial\Phi}{\partial\phi}\right)x-\left(\frac{1}{x^2+y^2}\frac{\partial\Phi}{\partial\lambda}\right)y$$

$$\ddot{y}=\left(\frac{1}{r}\frac{\partial\Phi}{\partial r}-\frac{z}{r^2\sqrt{x^2+y^2}}\frac{\partial\Phi}{\partial\phi}\right)y+\left(\frac{1}{x^2+y^2}\frac{\partial\Phi}{\partial\lambda}\right)x$$

$$\ddot{z}=\left(\frac{1}{r}\frac{\partial\Phi}{\partial r}\right)z+\left(\frac{\sqrt{x^2+y^2}}{r^2}\frac{\partial\Phi}{\partial\phi}\right)$$

The three partial derivatives of the geopotential with respect to r, ϕ, λ are given by

$$\frac{\partial \Phi}{\partial r} = -\frac{1}{r}\left(\frac{\mu}{r}\right)\sum_{n=2}^{N}\left(\frac{R}{r}\right)^{n}(n+1)\sum_{m=0}^{n}\left(C_n^m \cos m\lambda + S_n^m \sin m\lambda\right)P_n^m\left(\sin\phi\right)$$

$$\frac{\partial \Phi}{\partial \phi} = \left(\frac{\mu}{r}\right)\sum_{n=2}^{N}\left(\frac{R}{r}\right)^{n}\sum_{m=0}^{n}\left(C_n^m \cos m\lambda + S_n^m \sin m\lambda\right)\left[P_n^{m+1}\left(\sin\phi\right) - m\tan\phi\, P_n^m\left(\sin\phi\right)\right]$$

$$\frac{\partial \Phi}{\partial \lambda} = \left(\frac{\mu}{r}\right)\sum_{n=2}^{N}\left(\frac{R}{r}\right)^{n}\sum_{m=0}^{n} m\left(S_n^m \cos m\lambda - C_n^m \sin m\lambda\right)P_n^m\left(\sin\phi\right)$$

where

$R =$ radius of the Earth
$r =$ geocentric distance of the spacecraft
$S_n^m, C_n^m =$ harmonic coefficients
$\phi =$ geocentric latitude of the spacecraft $= \sin^{-1}\left(z/r\right)$
$\lambda =$ longitude of the spacecraft $= \alpha - \alpha_g$
$\alpha =$ right ascension of the spacecraft $= \tan^{-1}\left(r_y/r_x\right)$
$\alpha_g =$ right ascension of Greenwich

Right ascension is measured positive east of the vernal equinox, longitude is measured positive east of Greenwich, and latitude is positive above the Earth's equator and negative below.

For $m = 0$, the coefficients are called *zonal* terms, when $m = n$ the coefficients are *sectorial* terms, and for $n > m \neq 0$ the coefficients are called *tesseral* terms.

The Legendre polynomials with argument $\sin\phi$ are computed using recursion relationships given by:

$$P_n^0\left(\sin\phi\right) = \frac{1}{n}\left[\left(2n-1\right)\sin\phi\, P_{n-1}^0\left(\sin\phi\right) - \left(n-1\right)P_{n-2}^0\left(\sin\phi\right)\right]$$

$$P_n^n\left(\sin\phi\right) = \left(2n-1\right)\cos\phi\, P_{n-1}^{n-1}\left(\sin\phi\right), \qquad m \neq 0,\ m < n$$

$$P_n^m\left(\sin\phi\right) = P_{n-2}^m\left(\sin\phi\right) + \left(2n-1\right)\cos\phi\, P_{n-1}^{m-1}\left(\sin\phi\right), \qquad m \neq 0,\ m = n$$

where the first few associated Legendre functions are given by

$$P_0^0\left(\sin\phi\right) = 1, \quad P_1^0\left(\sin\phi\right) = \sin\phi, \quad P_1^1\left(\sin\phi\right) = \cos\phi$$

and $P_i^j = 0$ for $j > i$.

The trigonometric arguments are determined from expansions given by

$$\sin m\lambda = 2\cos\lambda \sin\left(m-1\right)\lambda - \sin\left(m-2\right)\lambda$$

$$\cos m\lambda = 2\cos\lambda \cos\left(m-1\right)\lambda - \cos\left(m-2\right)\lambda$$

$$m\tan\phi = \left(m-1\right)\tan\phi + \tan\phi$$

The true-of-date position vector required in the previous equations is computed according to

$$\mathbf{r}_{TOD} = \left[PN\right]\mathbf{r}_{EME2000}$$

where $\left[PN\right]$ is the combined precession-nutation matrix.

The east longitude required in the gravity model calculations is computed from the x and y components of the true-of-date position vector according to

$$\lambda = \tan^{-1}\left(r_y, r_x\right) - \alpha_g$$

where α_g is the apparent right ascension of Greenwich at the time of interest.

The true-of-date gravity vector is converted to the EME2000 system for use in the equations of motion using the transpose of the combined precession-nutation matrix as follows

$$\mathbf{a}_{EME2000} = \left[PN\right]^T \mathbf{a}_{TOD}$$

Point mass acceleration of the sun and moon

The acceleration contribution of the moon represented by a *point mass* is given by

$$\vec{a}_m\left(\vec{r},t\right) = -\mu_m\left(\frac{\vec{r}_{m-b}}{\left|\vec{r}_{m-b}\right|^3} + \frac{\vec{r}_{e-m}}{\left|\vec{r}_{e-m}\right|^3}\right)$$

where

$\mu_m =$ gravitational constant of the moon
$\vec{r}_{m-b} =$ position vector from the moon to the spacecraft
$\vec{r}_{e-m} =$ position vector from the Earth to the moon

Likewise, the acceleration contribution of the sun represented by a *point mass* is given by

$$\vec{a}_s\left(\vec{r},t\right) = -\mu_s\left(\frac{\vec{r}_{s-b}}{\left|\vec{r}_{s-b}\right|^3} + \frac{\vec{r}_{e-s}}{\left|\vec{r}_{e-s}\right|^3}\right)$$

where

$$\mu_s = \text{ gravitational constant of the sun}$$
$$\vec{r}_{s-b} = \text{ position vector from the sun to the spacecraft}$$
$$\vec{r}_{e-s} = \text{ position vector from the Earth to the sun}$$

To avoid numerical problems, use is made of Richard Battin's $f(q)$ function given by

$$f(q_k) = q_k \left[\frac{3 + 3q_k + q_k^2}{1 + \left(\sqrt{1+q_k}\right)^3} \right]$$

where

$$q_k = \frac{\mathbf{r}^T \left(\mathbf{r} - 2\mathbf{s}_k\right)}{\mathbf{s}_k^T \mathbf{s}_k}$$

The point-mass acceleration due to *n* gravitational bodies can now be expressed as

$$\ddot{\mathbf{r}} = -\sum_{k=1}^{n} \frac{\mu_k}{d_k^3} \left[\mathbf{r} + f(q_k)\mathbf{s}_k \right]$$

In these equations, $\mathbf{s}_k$ is the vector from the primary body to the secondary body, μ_k is the gravitational constant of the secondary body, and $\mathbf{d}_k = \mathbf{r} - \mathbf{s}_k$, where $\mathbf{r}$ is the position vector of the IRIS spacecraft relative to the primary body. The derivation of the $f(q)$ functions is described in Section 8.4 of "An Introduction to the Mathematics and Methods of Astrodynamics, Revised Edition", by Richard H. Battin, AIAA Education Series, 1999.

Flight path coordinates at the entry interface

This section describes the algorithms used to compute the following flight path coordinates at the entry interface.

$$r = \text{ geocentric radius}$$
$$V = \text{ speed}$$
$$\gamma = \text{ flight path angle}$$
$$\delta = \text{ geocentric declination}$$
$$\lambda = \text{ geographic longitude } (+\text{ east})$$
$$\psi = \text{ flight azimuth } (+\text{ clockwise from north})$$

The transformation of an Earth-centered inertial (ECI) position vector $\mathbf{r}_{ECI}$ to an Earth-centered fixed (ECF) position vector $\mathbf{r}_{ECF}$ is given by the following vector-matrix operation

$$\mathbf{r}_{ECF} = [\mathbf{T}]\mathbf{r}_{ECI}$$

where the elements of the transformation matrix $[\mathbf{T}]$ are given by

$$[\mathbf{T}] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and θ is the Greenwich apparent sidereal time at the moment of interest. Greenwich sidereal time is given by the following expression:

$$\theta = \theta_{g0} + \omega_e t$$

where θ_{g0} is the Greenwich sidereal time at 0 hours UTC, ω_e is the inertial rotation rate of the Earth, and t is the elapsed time since 0 hours UTC.

Finally, the flight path coordinates are determined from the ECF coordinates of the vehicle using following set of equations

$$r = \sqrt{r_{ECF}^2 + r_{ECF_y}^2 + r_{ECF}^2}$$

$$v = \sqrt{v_{ECF}^2 + v_{ECF_y}^2 + v_{ECF_z}^2}$$

$$\lambda = \tan^{-1}\left(r_{ECF_y}, r_{ECF_x}\right)$$

$$\delta = \sin^{-1}\left(\frac{r_{ECF_z}}{|\mathbf{r}_{ECF}|}\right)$$

$$\gamma = \sin^{-1}\left(-\frac{v_{R_z}}{|\mathbf{v}_R|}\right)$$

$$\psi = \tan^{-1}\left[v_{R_y}, v_{R_x}\right]$$

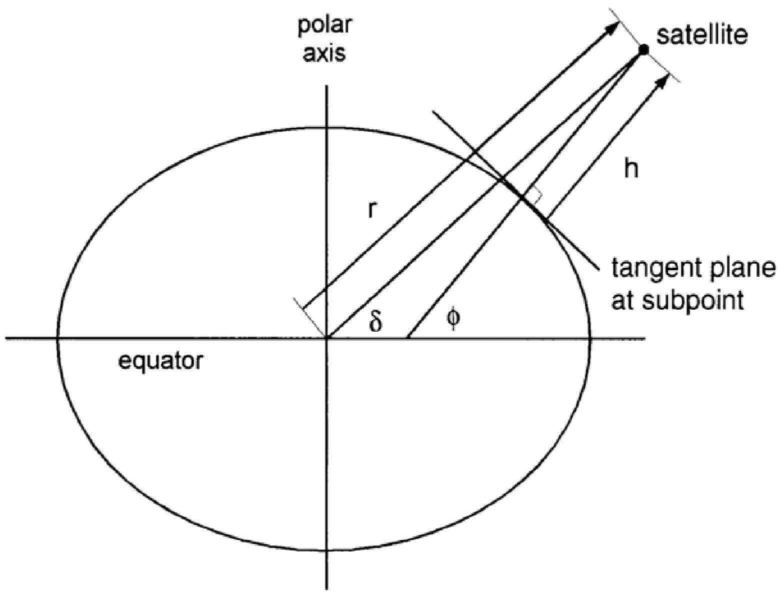
where

$$\mathbf{v}_R = \begin{bmatrix} -\sin\delta\cos\lambda & -\sin\delta\sin\lambda & \cos\delta \\ -\sin\lambda & \cos\lambda & 0 \\ -\cos\delta\cos\lambda & -\cos\delta\sin\lambda & -\sin\delta \end{bmatrix} \mathbf{v}_{ECF}$$

Please note that the two argument inverse tangent calculation is a four quadrant operation.

Geodetic altitude and latitude

These computer programs use a series solution to convert geocentric radius and declination to geodetic altitude and latitude. The following diagram illustrates the geometric relationship between geocentric and geodetic coordinates.



In this diagram, δ is the geocentric declination, ϕ is the geodetic latitude, r is the geocentric distance, and h is the geodetic altitude.

The exact mathematical relationship between geocentric and geodetic coordinates is given by the following system of two nonlinear equations

$$(c+h)\cos\phi - r\cos\delta = 0$$

$$(s+h)\sin\phi - r\sin\delta = 0$$

where the geodetic constants c and s are given by

$$c = \frac{r_{eq}}{\sqrt{1-\left(2f - f^2\right)\sin^2\phi}}$$

$$s = c\left(1-f\right)^2$$

and r_{eq} is the Earth equatorial radius and f is the flattening factor for the Earth.

In these computer programs, the geodetic latitude is determined using the following expansion in flattening factor:

$$\phi = \delta + \left(\frac{\sin 2\delta}{\rho}\right) f + \left[\left(\frac{1}{\rho^2} - \frac{1}{4\rho}\right)\sin 4\delta\right] f^2$$

The geodetic altitude is calculated from

$$\hat{h} = (\hat{r} - 1) + \left\{ \left(\frac{1 - \cos 2\delta}{2} \right) f + \left[\left(\frac{1}{4\rho} - \frac{1}{16} \right) (1 - \cos 4\delta) \right] f^2 \right\}$$

In these equations, ρ is the geocentric distance of the satellite, $\hat{h} = h / r_{eq}$ and $\hat{r} = \rho / r_{eq}$.