Relative Motion Between Two Earth Satellites

This document describes a MATLAB script that can be used to design and analyze relative motion trajectories between two Earth satellites in circular orbits. The algorithms in this section are based on the techniques described in the classic paper, “Terminal Guidance System for Satellite Rendezvous”, by W. H. Clohessy and R. S. Wiltshire, *Journal of the Aerospace Sciences*, Vol. 27, 1960.

In the following discussion the passive satellite is called the *target* and the active or maneuvering satellite is called the *chaser*. The state vector of the chaser satellite is defined with respect to a local vertical-local horizontal (LVLH) coordinate system centered at the target satellite.

The relationship between the chaser vehicle state vector $x, y, z, \dot{x}, \dot{y}$ and $\dot{z}$ at any time *t* to the initial state vector $x_0, y_0, z_0, \dot{x}_0, \dot{y}_0$ and $\dot{z}_0$ at time t_0 is given by the following state transition matrix for *unperturbed* relative motion:

$$\begin{Bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{Bmatrix} = \begin{bmatrix} 1 & -6\left(\omega t - \sin\omega t\right) & 0 & -3t + \frac{4}{\omega}\sin\omega t & -\frac{2}{\omega}\left(1 - \cos\omega t\right) & 0 \\ 0 & 4 - 3\cos\omega t & 0 & \frac{2}{\omega}\left(1 + \cos\omega t\right) & \frac{1}{\omega}\sin\omega t & 0 \\ 0 & 0 & \cos\omega t & 0 & 0 & \frac{1}{\omega}\sin\omega t \\ 0 & 6\omega\left(1 - \cos\omega t\right) & 0 & -3 + 4\cos\omega t & 2\sin\omega t & 0 \\ 0 & 3\omega\sin\omega t & 0 & -2\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & -\omega\sin\omega t & 0 & 0 & \cos\omega t \end{bmatrix} \begin{Bmatrix} x_0 \\ y_0 \\ z_0 \\ \dot{x}_0 \\ \dot{y}_0 \\ \dot{z}_0 \end{Bmatrix}$$

where $\omega = \sqrt{\mu / \left(r / r_e\right)^3}$ is the orbital rate of the target’s circular orbit with *r* equal to the radius of the target’s orbit and r_e equal to the radius of the Earth. In this equation, μ is the gravitational constant of the Earth.

The x, y and z position components of the chaser satellite as a function of time are given by the following three expressions:

$$x\left(t\right) = \left[x_0 - \frac{2\dot{y}_0}{\omega}\right] + \left[-3\dot{x}_0 - 6\omega y_0\right]t + \left[2\frac{\dot{y}_0}{\omega}\right]\cos\omega t + \left[4\frac{\dot{x}_0}{\omega} + 6y_0\right]\sin\omega t$$

$$y\left(t\right) = \left[2\frac{\dot{x}_0}{\omega} + 4y_0\right] + \left[-2\frac{\dot{x}_0}{\omega} - 3y_0\right]\cos\omega t + \left[\frac{\dot{y}_0}{\omega}\right]\sin\omega t$$

$$z\left(t\right) = z_0\cos\omega t + \frac{\dot{z}_0}{\omega}\sin\omega t$$

grmotion.m – relative motion of two Earth satellites in circular orbits

This MATLAB script calculates and graphically displays the relative Keplerian motion between two satellites in circular Earth orbits. This script provides the following user options:

- user input of initial conditions
- calculate and display synchronous orbit
- calculate and display two impulse rendezvous orbit

This MATLAB script displays two-dimensional motion in the x-y or orbit plane.

User input of initial conditions

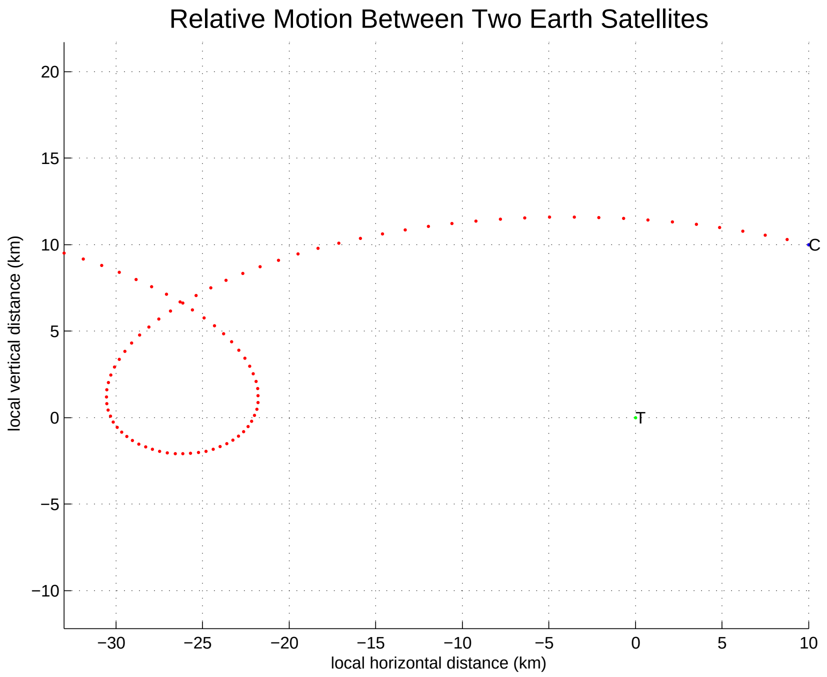
This program option allows the user to input the initials conditions of the chaser vehicle relative to the target. The user can graphically display the relative trajectory for either one orbital period or a user-defined duration.

The following is a typical user interaction with this script and this option.

```
      graphics display of relative motion


           relative motion menu

    <1> user input of initial conditions

    <2> calculate and display synchronous orbit

    <3> calculate and display rendezvous orbit

    selection (1, 2 or 3)

   ? 1


   please input the altitude of the target satellite (kilometers)
   ? 350

   please input the initial x-position of the chaser satellite (kilometers)
   ? 10

   please input the initial y-position of the chaser satellite (kilometers)
   ? 10


   please input delta-vx of the chaser satellite (meters/second)
   ? -3

   please input delta-vy of the chaser satellite (meters/second)
   ? 5
```

```
           user-defined orbit

target altitude       350.0000  kilometers
chaser x distance      10.0000  kilometers
chaser y distance      10.0000  kilometers

vx prior to maneuver   -17.1600  meters/second
vy prior to maneuver     0.0255  meters/second

maneuver delta-vx       -3.0000  meters/second
maneuver delta-vy        5.0000  meters/second
total delta-v            5.8310  meters/second


the orbital period is    91.5382  minutes

     simulation time menu

 <1> user input of simulation time

 <2> simulate for one orbital period

selection (1 or 2)
? 2

please input the plot step size (minutes)
? 1
```

The following is the companion graphics display for this example. The orbital motion of the target satellite is to the right. The target satellite is labeled with the letter T and the chaser satellite is labeled with the letter C. The chaser trajectory is displayed at the time interval input by the user.



The relative motion trajectory of the chaser spacecraft is a “drifting” ellipse with its center located at (c, d) where

$$c = x_0 - 2\frac{\dot{y}_0}{\omega} - \left(3\dot{x}_0 + 6\omega y_0\right)t$$

$$d = \frac{2\dot{x}_0}{\omega} + 4y_0$$

The semimajor axis of this dynamic ellipse is given by

$$a = 2\left\{\left(-2\frac{\dot{x}_0}{\omega} - 3y_0\right)^2 + \left(\frac{\dot{y}_0}{\omega}\right)^2\right\}$$

and the semiminor axis is equal to $a/2$.

Synchronous orbit

The initial velocity components required for a chaser vehicle to be synchronous or “co-orbital” with a target vehicle located in a user-defined circular orbit are given by

$$\dot{x}_{0_s} = -2\omega y_0$$

$$\dot{y}_{0_s} = 0$$

The initial velocity components for any initial x_0 and y_0 position components are given by the following two expressions:

$$\dot{x}_0 = -\frac{3}{2}\omega y_0$$

$$\dot{y}_0 = \frac{\frac{3}{2}\omega x_0 y_0}{\frac{r_{eq}}{r} + y_0}$$

Therefore, the components of the initial velocity increment for a synchronous orbit are given by

$$\Delta V_x = \dot{x}_{0_s} - \dot{x}_0$$

$$\Delta V_y = \dot{y}_{0_s} - \dot{y}_0$$

The relative motion trajectory is an ellipse with its center located at $(c,0)$ where

$$c = x_0 - 2\frac{\dot{y}_0}{\omega}$$

The semimajor axis of this ellipse is given by

$$a = 2\sqrt{y_0^2 + \left[\frac{\dot{y}_0}{\omega}\right]^2}$$

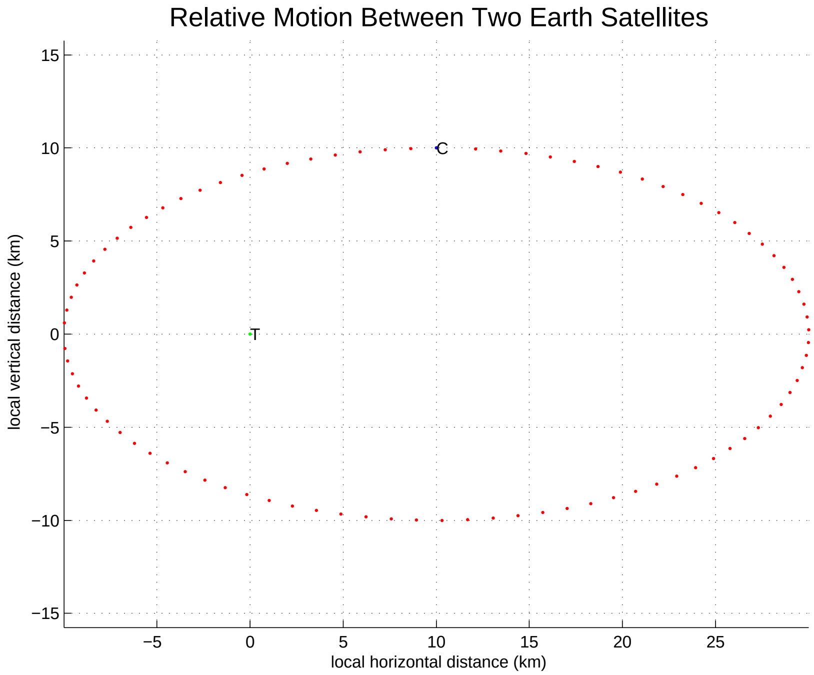
and the semiminor axis is equal to $a/2$.

The following is a typical user interaction and data output for this program option.

```
graphics display of relative motion


        relative motion menu

 <1> user input of initial conditions

 <2> calculate and display synchronous orbit

 <3> calculate and display rendezvous orbit

 selection (1, 2 or 3)

? 2

please input the altitude of the target satellite (kilometers)
? 350

please input the initial x-position of the chaser satellite (kilometers)
? 10

please input the initial y-position of the chaser satellite (kilometers)
? 10

           synchronous orbit

target altitude       350.0000  kilometers
chaser x distance      10.0000  kilometers
chaser y distance      10.0000  kilometers

vx prior to maneuver   -17.1600  meters/second
vy prior to maneuver     0.0255  meters/second

maneuver delta-vx       -5.7200  meters/second
maneuver delta-vy       -0.0255  meters/second
total delta-v            5.7201  meters/second
```

```
the orbital period is    91.5382  minutes


     simulation time menu

 <1> user input of simulation time

 <2> simulate for one orbital period

selection (1 or 2)
? 2

please input the plot step size (minutes)
? 1
```

The following is the companion graphics display of a synchronous orbit. The orbital motion of the target satellite is to the right. The target satellite is labeled with the letter T and the chaser satellite is labeled with the letter C. The chaser trajectory is displayed at the time interval input by the user.



Two impulse rendezvous orbit

Orbital rendezvous is the process of bringing a chaser vehicle from some initial location to a final location with zero relative velocity in a specified transfer time. This type of orbit transfer involves an initial maneuver that starts the transfer and a second maneuver that stops the chaser spacecraft at the final location. This program option of the `grmotion` script calculates the magnitude and direction of these two *impulsive* maneuvers and graphically displays the transfer trajectory.

The initial velocity components required for a rendezvous orbit are given by

$$\dot{x}_{0_{tpi}} = \frac{14y_0\left(1-\cos\omega t_r\right)-\left(6y_0\omega t_r - x_0\right)\sin\omega t_r}{t_r\left[3\sin\omega t_r - \dfrac{8}{\omega t_r}\left(1-\cos\omega t_r\right)\right]}$$

$$\dot{y}_{0_{tpi}} = \frac{-y_0\left(3\omega t_r\cos\omega t_r - 4\sin\omega t_r\right)-2x_0\left(1-\cos\omega t_r\right)}{t_r\left[3\sin\omega t_r - \dfrac{8}{\omega t_r}\left(1-\cos\omega t_r\right)\right]}$$

where t_r is the transfer time.

The components of the *terminal phase initiation* ΔV required to start the orbital rendezvous are determined from the following equations

$$\Delta V_x = \dot{x}_{0_{tpi}} - \dot{x}_0$$

$$\Delta V_y = \dot{y}_{0_{tpi}} - \dot{y}_0$$

where $\dot{x}_0$ and $\dot{y}_0$ are the x and y velocity components of the chaser vehicle prior to this impulsive maneuver.

The components of the ΔV required to brake the vehicle at the target are given by

$$\dot{x}_b\left(t_r\right) = \left(-3\dot{x}_{tpi} - 6\omega y_0\right) + \left(-2\dot{y}_{tpi}\right)\sin\omega t_r + \left(4\dot{x}_{tpi} + 6\omega y_0\right)\cos\omega t_r$$

$$\dot{y}_b\left(t_r\right) = \left(2\dot{x}_{tpi} + 3y_0\omega\right)\sin\omega t_r + \left(\dot{y}_{tpi}\right)\cos\omega t_r$$

The following is a typical user interaction with this MATLAB script and this program option.

```
        relative motion menu

 <1> user input of initial conditions

 <2> calculate and display synchronous orbit

 <3> calculate and display rendezvous orbit

 selection (1, 2 or 3)

? 3

please input the altitude of the target satellite (kilometers)
? 300
```

```
please input the initial x-position of the chaser satellite (kilometers)
? 50

please input the initial y-position of the chaser satellite (kilometers)
? -100

please input the rendezvous time (minutes)
? 120

           rendezvous orbit

target altitude        300.0000  kilometers
chaser x distance       50.0000  kilometers
chaser y distance     -100.0000  kilometers

time to rendezvous     120.0000  minutes

vx prior to tpi        173.5309  meters/second
vy prior to tpi         -1.3190  meters/second

tpi delta-vx            94.6752  meters/second
tpi delta-vy          -179.0340  meters/second
tpi delta-v            202.5255  meters/second

braking delta-vx        36.8316  meters/second
braking delta-vy       250.9074  meters/second
braking delta-v        253.5964  meters/second

total delta-v          456.1219  meters/second

please input the plot step size (minutes)
? 1
```