Optimal Impulsive Orbital Transfer

This document describes an algorithm and MATLAB script called `oota.m` that can be used to determine optimum one and two impulse orbital transfers between *non-coplanar* circular and elliptical orbits. The method is general and the initial and final orbits need not be coapsidal. The algorithm is based on the orbit transfer and rendezvous work of Gary McCue, Gentry Lee and David Bender, described in "Numerical Investigation of Minimum Impulse Orbital Transfer", *AIAA Journal*, **3**, 2328-2334 (1963); and "An Analysis of Two-Impulse Orbital Transfer", *AIAA Journal*, **2**, 1767-1773 (1964).

The numerical solution of this classic astrodynamic problem involves a combination of one-dimensional root-finding using Brent's method and multi-dimensional constrained minimization using the SNOPT nonlinear programming (NLP) algorithm. This script also performs a graphical primer vector analysis of two impulse orbital transfers.

Data file format

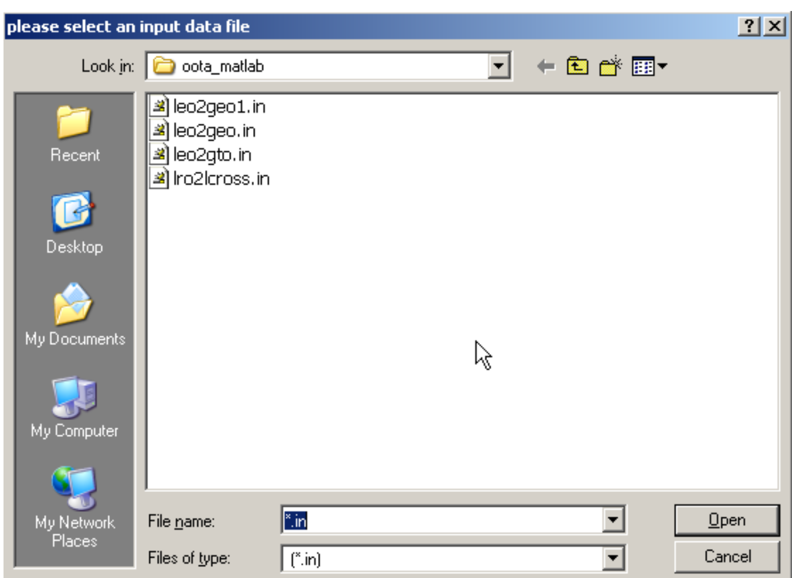
The script reads a simple ASCII data file that defines the initial and final orbits. The following is a typical data file named `leo2geo.in` for this application. This example solves the problem of two impulse, non-coplanar orbital transfer from a typical low altitude Earth orbit (LEO) to a geosynchronous Earth orbit (GEO). The annotation text in this file can be modified but should not be deleted because the routine that reads this data expects to find exactly 59 lines of text and numeric information. The first two data items define the gravitational constant and radius of the central body. Please note the units and valid range for each input.

```
******************************************
* input data file for oota.m Matlab script
* LEO-to-GEO orbital transfer
* filename ==> leo2geo.in
******************************************

central body gravitational constant (km**3/sec**2)
398600.5

central body radius (kilometers)
6378.14

*************
initial orbit
*************

semimajor axis (kilometers)
(semimajor axis > 0)
6653.14

orbital eccentricity (non-dimensional)
(0 <= eccentricity < 1)
0

orbital inclination (degrees)
(0 <= inclination <= 180)
28.5

argument of perigee (degrees)
```

```
(0 <= argument of perigee <= 360)
0

right ascension of the ascending node (degrees)
(0 <= raan <= 360)
0

***********
final orbit
***********

semimajor axis (kilometers)
(semimajor axis > 0)
42166.2355

orbital eccentricity (non-dimensional)
(0 <= eccentricity < 1)
0

orbital inclination (degrees)
(0 <= inclination <= 180)
5

argument of perigee (degrees)
(0 <= argument of perigee <= 360)
0

right ascension of the ascending node (degrees)
(0 <= raan <= 360)
0
```

Running the script

When the `oota` script is started, the software will display a screen similar to the following which allows the user to select a data file for processing.



The file type defaults to names with a `*.in` filename extension. However, you can select any `oota.m` compatible ASCII data file by selecting the Files of type: field or by typing the name of the file directly in the File name: field.

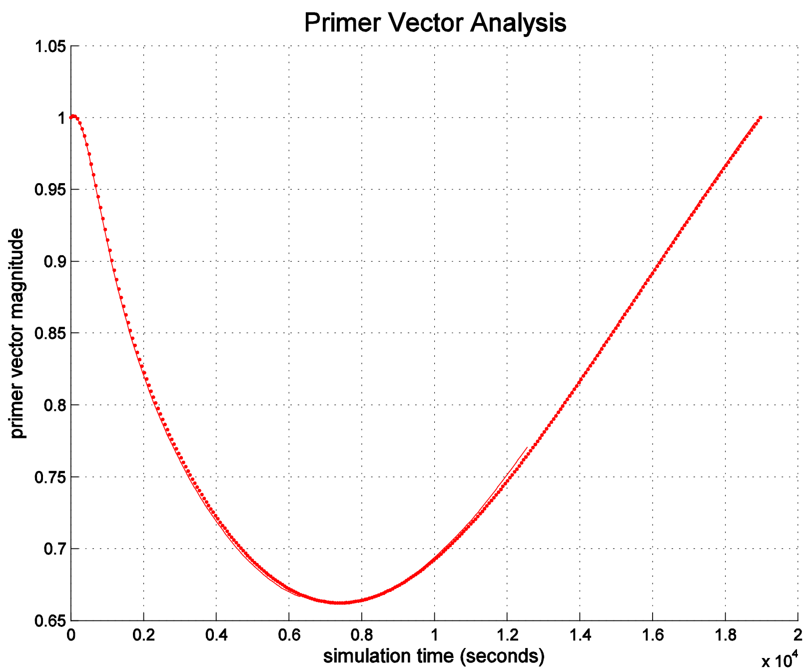
Optimal solution

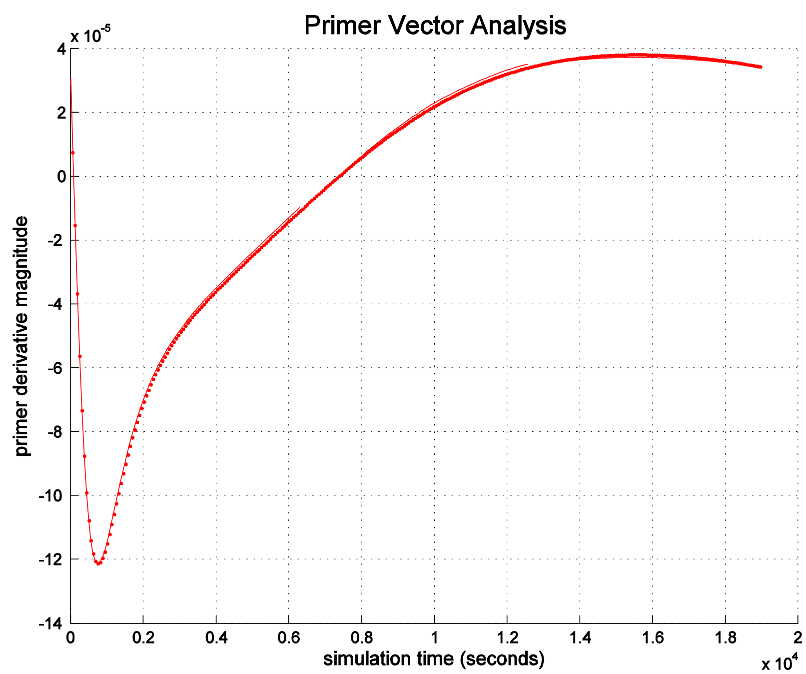
The following is the output created by the `oota.m` script for a LEO-to-GEO orbit transfer example. The pitch and yaw angles for each maneuver are computed and displayed in the local-vertical-local horizontal (LVLH) coordinate system.

```
initial orbit - first impulse

        sma (km)              eccentricity          inclination (deg)         argper (deg)
 +6.65314000000000e+003  +1.11022302462516e-016  +2.85000000000000e+001  +0.00000000000000e+000

       raan (deg)          true anomaly (deg)          arglat (deg)           period (min)
 +0.00000000000000e+000  +9.70885466425371e-002  +9.70885466425371e-002  +9.00118508344800e+001

delta-v LVLH pitch angle          0.1252 degrees

delta-v LVLH yaw angle            8.0023 degrees

transfer orbit - first impulse

        sma (km)              eccentricity          inclination (deg)         argper (deg)
 +2.44096963497253e+004  +7.27438731116838e-001  +2.65778847608706e+001  +3.19063555037724e-002

       raan (deg)          true anomaly (deg)          arglat (deg)           period (min)
 +3.59992721627361e+002  +7.16368087943620e-002  +1.03543164298134e-001  +6.32562240174735e+002

transfer orbit - second impulse

        sma (km)              eccentricity          inclination (deg)         argper (deg)
 +2.44096963485282e+004  +7.27438731100966e-001  +2.65778847608706e+001  +3.19063555087684e-002

       raan (deg)          true anomaly (deg)          arglat (deg)           period (min)
 +3.59992721627361e+002  +1.79966368763508e+002  +1.79998275119016e+002  +6.32562240128201e+002

delta-v LVLH pitch angle         -0.0853 degrees

delta-v LVLH yaw angle          317.9581 degrees

final orbit - second impulse

        sma (km)              eccentricity          inclination (deg)         argper (deg)
 +4.21662355000000e+004  +1.11022305746105e-016  +5.00000000000000e+000  +0.00000000000000e+000

       raan (deg)          true anomaly (deg)          arglat (deg)           period (min)
 +0.00000000000000e+000  +1.79991145325018e+002  +1.79991145325018e+002  +1.43617361385198e+003

ECI delta-v vectors and magnitudes

delta-v1x               1.2434 meters/second
delta-v1y            2295.8835 meters/second
delta-v1z             858.2926 meters/second

delta-v1             2451.0710 meters/second

delta-v2x               2.2701 meters/second
delta-v2y           -1627.3466 meters/second
delta-v2z             450.2037 meters/second

delta-v2             1688.4743 meters/second

total delta-v        4139.5453 meters/second
```

```
  transfer time         18976.8672 seconds
```

The graphical primer vector analysis for this example is shown below. These plots illustrate the behavior of the scalar magnitudes of the primer vector and its derivative as a function of the orbit transfer time.

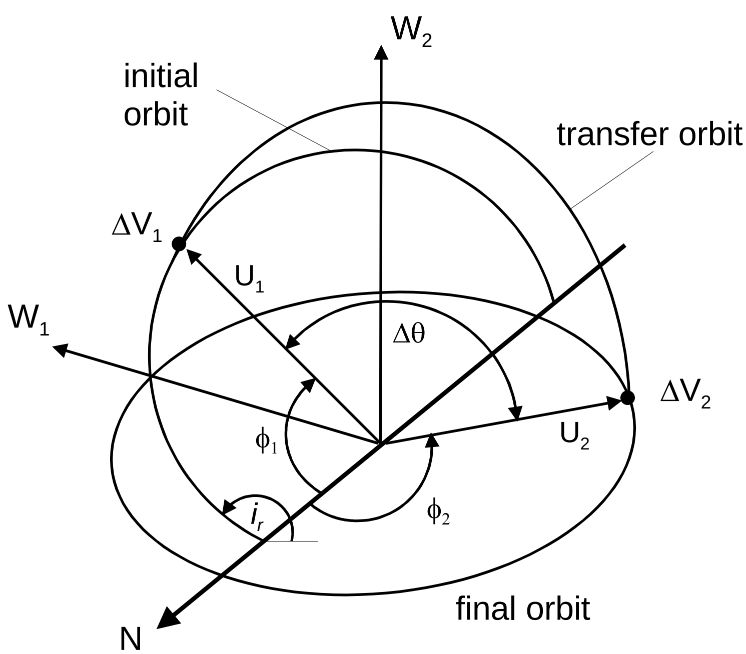




Technical Discussion

The solution to this important astrodynamics problem is formulated in a *reference coordinate* system. The fundamental reference plane of this coordinate system is the final orbit plane and the x-axis is aligned with the intersection of the planes of the initial and final orbits. The z-axis of this system is aligned with the angular momentum vector of the final orbit and the y-axis completes this orthogonal coordinate system. In the equations which follow, elements of the initial transfer orbit have a subscript of *1* and elements of the final orbit a subscript of *2*. Elements of the transfer orbit will have a subscript of *t*.

The following diagram illustrates the geometry of two impulse orbital transfer. The relative inclination between the initial and final orbit planes is i_r and D∠ is the transfer angle which is the angle from the first and second impulse measured in the plane of the transfer orbit. **N** corresponds to the x-axis, $\mathbf{W}_1$ is in the direction of the initial orbit angular momentum vector, and $\mathbf{W}_2$ is in the direction of the angular momentum vector of the final orbit.



The independent variables for this problem are f_1, f_2 and p_t, where f_1 is the angle from the **N** axis to the first impulse as measured in the initial orbit plane, f_2 is the angle from the **N** axis to the second impulse as measured in the final orbit plane, and p_t is the semiparameter of the transfer orbit. The expression for **N** is as follows:

$$\mathbf{N} = \frac{\mathbf{W}_2 \times \mathbf{W}_1}{\left|\mathbf{W}_2 \times \mathbf{W}_1\right|}$$

where $\mathbf{W}_1$ can be calculated with

$$\mathbf{W}_1 = \begin{bmatrix} 0 \\ -\sin i_r \\ \cos i_r \end{bmatrix}$$

and $\mathbf{W}_2$ is determined from

$$\mathbf{W}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The relative inclination between the initial and final orbit planes is determined from

$$i_r = \cos^{-1}\left(\mathbf{w}_1 \bullet \mathbf{w}_2\right)$$

where $\mathbf{w}_1$ is the ECI unit angular momentum vector of the initial orbit given by

$$\mathbf{w}_1 = \begin{bmatrix} \sin\Omega_1 \sin i_1 \\ -\cos\Omega_1 \sin i_1 \\ \cos i_1 \end{bmatrix}$$

and $\mathbf{w}_2$ is the ECI unit angular momentum vector of the final orbit given by

$$\mathbf{w}_2 = \begin{bmatrix} \sin\Omega_2 \sin i_2 \\ -\cos\Omega_2 \sin i_2 \\ \cos i_2 \end{bmatrix}$$

The unit position vector at the first impulse in the reference coordinate system is

$$\mathbf{U}_1 = \begin{bmatrix} \cos\phi_1 \\ \sin\phi_1 \cos i_r \\ \sin\phi_1 \sin i_r \end{bmatrix}$$

and the unit position vector of the second impulse, also in the reference coordinate system, is determined from

$$\mathbf{U}_2 = \begin{bmatrix} \cos\phi_2 \\ \sin\phi_2 \\ 0 \end{bmatrix}$$

The transfer angle can be computed from the following dot product:

$$\Delta\theta = \cos^{-1}\left(\mathbf{U}_1 \bullet \mathbf{U}_2\right)$$

The minimum and maximum bounds on the semiparameter of the transfer orbit can be determined from the following two expressions:

$$p_{\min} = \frac{r_1 r_2 - \mathbf{r}_1 \bullet \mathbf{r}_2}{r_1 + r_2 + \sqrt{2\left(r_1 r_2 + \mathbf{r}_1 \bullet \mathbf{r}_2\right)}}$$

$$p_{\max} = \frac{r_1 r_2 - \mathbf{r}_1 \bullet \mathbf{r}_2}{r_1 + r_2 - \sqrt{2\left(r_1 r_2 + \mathbf{r}_1 \bullet \mathbf{r}_2\right)}}$$

The partial derivative of the total required ΔV with respect to the semiparameter of the transfer orbit is as follows:

$$\frac{\partial V_t}{\partial p_t} = \frac{1}{2 p_t}\left(\frac{\Delta\mathbf{V}_1 \bullet \left(\mathbf{V} - z\mathbf{U}_1\right)}{\left|\Delta\mathbf{V}_1\right|} - \frac{\Delta\mathbf{V}_2 \bullet \left(\mathbf{V} + z\mathbf{U}_2\right)}{\left|\Delta\mathbf{V}_2\right|}\right)$$

Part of the optimal orbital transfer solution involves finding the value of p_t which lies between $p_{\min}$ and $p_{\max}$ and makes this partial derivative expression equal to zero.

The DV vectors in the reference coordinate system are given by the following two expressions

$$\Delta\mathbf{V}_1 = \pm\left(\mathbf{V} + z\mathbf{U}_1\right) - \mathbf{V}_1$$

$$\Delta\mathbf{V}_2 = \mathbf{V}_2 \mp \left(\mathbf{V} - z\mathbf{U}_2\right)$$

where the upper sign in these two equations corresponds to the short transfer and

$$z = \sqrt{\frac{\mu}{p}}\tan\frac{\Delta\theta}{2}$$

with

$$\mathbf{V} = \sqrt{\mu p_t}\,\frac{\left(\mathbf{r}_2 - \mathbf{r}_1\right)}{\left|\mathbf{r}_1 \times \mathbf{r}_2\right|}$$

The velocity vector of the satellite prior to the first impulse with respect to the reference coordinate system is calculated from

$$\mathbf{V}_1 = \sqrt{\frac{\mu}{p_1}}\mathbf{W}_1 \times \left(\mathbf{e}_1 + \mathbf{U}_1\right)$$

and prior to the second impulse it is given by:

$$\mathbf{V}_2 = \sqrt{\frac{\mu}{p_2}}\mathbf{W}_2 \times \left(\mathbf{e}_2 + \mathbf{U}_2\right)$$

In these expressions $\mathbf{e}_1$ is the reference coordinate system eccentricity vector of the initial orbit is given by

$$\mathbf{e}_1 = e_1 \begin{bmatrix} \cos\omega_1 \\ \sin\omega_1 \cos i_r \\ \sin\omega_1 \sin i_r \end{bmatrix}$$

and $\mathbf{e}_2$ is the eccentricity of the final orbit defined by

$$\mathbf{e}_2 = e_2 \begin{bmatrix} \cos\omega_2 \\ \sin\omega_2 \\ 0 \end{bmatrix}$$

where e_1 and e_2 are the scalar eccentricity of the initial and final orbits, respectively.

The total energy required for the orbit transfer is given by

$$\Delta V = \left|\Delta \mathbf{V}_1\right| + \left|\Delta \mathbf{V}_2\right|$$

In terms of the scalar components of the two $\Delta V's$, the total ΔV required is

$$\Delta V = \sqrt{\Delta V_{1x}^2 + \Delta V_{1y}^2 + \Delta V_{1x}^2} + \sqrt{\Delta V_{2x}^2 + \Delta V_{2y}^2 + \Delta V_{2x}^2}$$

This is the scalar quantity we want to minimize.

Eventually, we want to convert the reference coordinate system solution to ECI vectors and then to classical orbital elements. The transformation of an ECI position or velocity vector $\mathbf{X}_{eci}$ to its corresponding reference coordinate system companion $\mathbf{X}_{rcs}$ is given by the following matrix-vector multiplication:

$$\mathbf{X}_{eci} = \left[\mathbf{T}\right]\mathbf{X}_{rcs}$$

The conversion of a vector in the reference coordinate system to its corresponding ECI vector involves the transpose of this matrix as follows:

$$\mathbf{X}_{rcs} = \left[\mathbf{T}\right]\mathbf{X}_{eci}$$

The elements of the reference coordinate system-to-ECI transformation matrix $\left[\mathbf{T}\right]$ are given by the following nine expressions:

$$
\begin{aligned}
T_{11} &= \cos\Omega_2 \cos\phi - \sin\Omega_2 \cos i_2 \sin\phi \\
T_{12} &= -\cos\Omega_2 \sin\phi - \sin\Omega_2 \cos i_2 \cos\phi \\
T_{13} &= \sin\Omega_2 \cos i_2 \\
T_{21} &= \sin\Omega_2 \cos\phi + \cos\Omega_2 \cos i_2 \sin\phi \\
T_{22} &= -\sin\Omega_2 \sin\phi + \cos\Omega_2 \cos i_2 \cos\phi \\
T_{23} &= -\cos\Omega_2 \sin i_2 \\
T_{31} &= \sin i_2 \sin\phi \\
T_{32} &= \sin i_2 \cos\phi \\
T_{33} &= \cos i_2
\end{aligned}
$$

where

$$\phi = -\cos^{-1}\left(\mathbf{N} \bullet \mathbf{U}\right)\operatorname{sign}\left(N_z\right)$$

and

$$\mathbf{U} = \begin{bmatrix} \cos\Omega_2 \\ \sin\Omega_2 \\ 0 \end{bmatrix}$$

The position vector of the initial and transfer orbits at the first impulse in the reference coordinate system is

$$\mathbf{r}_1 = \left(\frac{p_1}{1 + e_1 \cos\left(\phi_1 - \omega_1\right)}\right)\mathbf{U}_1$$

and the position vector of the transfer and final orbit at the second impulse is

$$\mathbf{r}_2 = \left(\frac{p_2}{1 + e_2 \cos\left(\phi_2 - \omega_2\right)}\right)\mathbf{U}_2$$

In these equations the arguments of perigee ω_1 and w_2 are with respect to the reference coordinate system. They can be determined with the following three equations:

$$\omega_{rcs} = \left[\mathbf{T}\right]^T \omega_{eci}$$

$$\omega_1 = \cos^{-1}\left(\omega_x\right)$$

$$\omega_2 = \tan^{-1}\left(\omega_y, \omega_z\right)$$

where the inverse tangent calculation here is a four quadrant operation.

The ECI argument of perigee vectors at each impulse are given by

$$\omega_{eci_1} = \begin{bmatrix} \cos\omega_1 \cos\Omega_1 - \sin\omega_1 \sin\Omega_1 \cos i_1 \\ \cos\omega_1 \sin\Omega_1 + \sin\omega_1 \cos\Omega_1 \cos i_1 \\ \sin\omega_1 \sin i_1 \end{bmatrix}$$

and

$$\omega_{eci_2} = \begin{bmatrix} \cos\omega_2 \cos\Omega_2 - \sin\omega_2 \sin\Omega_2 \cos i_2 \\ \cos\omega_2 \sin\Omega_2 + \sin\omega_2 \cos\Omega_2 \cos i_2 \\ \sin\omega_2 \sin i_2 \end{bmatrix}$$

where all the orbital elements in these two equations are with respect to the ECI coordinate system.

The semiparameter of the initial orbit can be determined from

$$p_1 = a_1\left(1 - e_1^2\right)$$

and the semiparameter of the final orbit is given by

$$p_2 = a_2\left(1 - e_2^2\right)$$

where a_1 and a_2 are the semimajor axes of the initial and final orbits, respectively.

The transfer orbit velocity vectors prior to the first and second impulses in the reference coordinate system are calculated from the next two equations:

$$\mathbf{V}_{T_1} = \mathbf{V} + z\mathbf{U}_1$$

$$\mathbf{V}_{T_2} = \mathbf{V} - z\mathbf{U}_2$$

The transfer orbit position and velocity vectors can be transformed into the ECI coordinate system using the transpose of the $[\mathbf{T}]$ matrix as described above, and then converted to classical orbital elements.

Primer Vector Analysis

This section summarizes the primer vector analysis included with this MATLAB script. The term primer vector was invented by Derek F. Lawden and represents the adjoint vector for velocity. A technical discussion about primer theory can be found in Lawden's classic text, *Optimal Trajectories for Space Navigation*, Butterworths, London, 1963. Another excellent resource is "Primer Vector Theory and Applications", Donald J. Jezewski, NASA TR R-454, November 1975, along with "Optimal, Multi-burn, Space Trajectories", also by Jezewski.

As shown by Lawden, the following four necessary conditions must be satisfied in order for an impulsive orbital transfer to be *locally optimal*:

(1) the primer vector and its first derivative are everywhere continuous

(2) whenever a velocity impulse occurs, the primer is a unit vector aligned with the impulse and has unit magnitude $\left(\mathbf{p} = \hat{\mathbf{p}} = \hat{\mathbf{u}}_T \text{ and } \|\mathbf{p}\| = 1\right)$

(3) the magnitude of the primer vector may not exceed unity on a coasting arc $\left(\|\mathbf{p}\| = p \le 1\right)$

(4) at all interior impulses (not at the initial or final times) $\mathbf{p} \bullet \dot{\mathbf{p}} = 0$; therefore, $d\|\mathbf{p}\|/dt = 0$ at the intermediate impulses

Furthermore, the scalar magnitudes of the primer vector derivative at the initial and final impulses provide information about how to improve the nominal transfer trajectory by changing the endpoint times and/or moving the impulse times. These four cases for non-zero slopes are summarized as follows;

- If $\dot{p}_0 > 0$ and $\dot{p}_f < 0 \rightarrow$ perform an initial coast before the first impulse and add a final coast after the second impulse
- If $\dot{p}_0 > 0$ and $\dot{p}_f > 0 \rightarrow$ perform an initial coast before the first impulse and move the second impulse to a later time
- If $\dot{p}_0 < 0$ and $\dot{p}_f < 0 \rightarrow$ perform the first impulse at an earlier time and add a final coast after the second impulse
- If $\dot{p}_0 < 0$ and $\dot{p}_f > 0 \rightarrow$ perform the first impulse at an earlier time and move the second impulse to a later time

The primer vector analysis of a two impulse orbital transfer involves the following steps.

First partition the two-body state transition matrix as follows:

$$\Phi\left(t, t_0\right) = \begin{bmatrix} \dfrac{\partial \mathbf{r}}{\partial \mathbf{r}_0} & \dfrac{\partial \mathbf{r}}{\partial \mathbf{v}_0} \\ \dfrac{\partial \mathbf{v}}{\partial \mathbf{r}_0} & \dfrac{\partial \mathbf{v}}{\partial \mathbf{v}_0} \end{bmatrix} = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} = \begin{bmatrix} \Phi_{rr} & \Phi_{rv} \\ \Phi_{vr} & \Phi_{vv} \end{bmatrix}$$

where

$$\Phi_{11} = \left[\frac{\partial \mathbf{r}}{\partial \mathbf{r}_0}\right] = \begin{bmatrix} \partial x / \partial x_0 & \partial x / \partial y_0 & \partial x / \partial z_0 \\ \partial y / \partial x_0 & \partial y / \partial y_0 & \partial y / \partial z_0 \\ \partial z / \partial x_0 & \partial z / \partial y_0 & \partial z / \partial z_0 \end{bmatrix}$$

and so forth.

The value of the primer vector at any time *t* along a two body trajectory is given by

$$\mathbf{p}(t) = \Phi_{11}(t,t_0)\mathbf{p}_0 + \Phi_{12}(t,t_0)\dot{\mathbf{p}}_0$$

and the value of the primer vector derivative is

$$\dot{\mathbf{p}}(t) = \Phi_{21}(t,t_0)\mathbf{p}_0 + \Phi_{22}(t,t_0)\dot{\mathbf{p}}_0$$

which can also be expressed as

$$\begin{Bmatrix}\mathbf{p}\\ \dot{\mathbf{p}}\end{Bmatrix} = \Phi(t,t_0)\begin{Bmatrix}\mathbf{p}_0\\ \dot{\mathbf{p}}_0\end{Bmatrix}$$

The primer vector boundary conditions at the initial and final impulses are as follows:

$$\mathbf{p}(t_0) = \mathbf{p}_0 = \frac{\Delta\mathbf{V}_0}{\left|\Delta\mathbf{V}_0\right|}$$

$$\mathbf{p}(t_f) = \mathbf{p}_f = \frac{\Delta\mathbf{V}_f}{\left|\Delta\mathbf{V}_f\right|}$$

These two conditions illustrate that at the locations of velocity impulses, the primer vector is a unit vector in the direction of the impulses.

The value of the primer vector derivative at the initial time is

$$\dot{\mathbf{p}}(t_0) = \dot{\mathbf{p}}_0 = \Phi_{12}^{-1}(t_f,t_0)\left\{\mathbf{p}_f - \Phi_{11}(t_f,t_0)\mathbf{p}_0\right\}$$

provided the Φ_{12} sub-matrix is non-singular.

The scalar magnitude of the derivative of the primer vector can be determined from

$$\frac{d\left\|\mathbf{p}\right\|}{dt} = \frac{d}{dt}(\mathbf{p}\bullet\mathbf{p})^2 = \frac{\dot{\mathbf{p}}\bullet\mathbf{p}}{\left\|\mathbf{p}\right\|}$$

SNOPT algorithm implementation

This section provides details about the part of the `oota.m` MATLAB script that solves this nonlinear programming (NLP) problem using the SNOPT 6.0 algorithm. In this classic trajectory optimization problem, the true anomalies of the initial and final orbits are the *control variables* and the total ΔV is the *objective function* or *performance index*.

MATLAB versions of SNOPT 6.0 for several computer platforms can be found at Professor Philip Gill's web site which is located at http://scicomp.ucsd.edu/~peg/. A PDF and Postscript version of the SNOPT user's manual is available at Professor Gill's website.

The SNOPT algorithm requires an initial guess for the control variables. For this problem they are given by

```
% control variables initial guess

xg(1) = 0.0;

xg(2) = 180.0 * dtr;

xg = xg';
```

where `xg(1)` is the true anomaly guess for the initial orbit delta-v and `xg(2)` is the true anomaly guess for the final orbit delta-v.

The algorithm also requires lower and upper bounds for the control variables. These are coded as follows:

```
% bounds on control variables

xlwr(1) = 0.0;
xupr(1) = 2.0 * pi;

xlwr(2) = 0.0;
xupr(2) = 2.0 * pi;


xlwr = xlwr';
xupr = xupr';

xlwr = xlwr';
xupr = xupr';
```

The algorithm also requires lower and upper bounds on the objective function. For this problem these bounds are defined by the following MATLAB statements:

```
% bounds on objective function

flow(1) = 0.0d0;

fupp(1) = +Inf;
```

The actual call to the SNOPT MATLAB interface function is as follows

```
[x, f, inform, xmul, fmul] = snopt(xg, xlwr, xupr, flow, fupp, 'ootafun1');
```

where `ootafun1` is the name of the MATLAB function that computes the current value of the objective function. The vector `x` is the optimal solution for the controls returned by SNOPT and `f` is the value of the objective function at the optimum.