Shadow Conditions of Earth Satellites

This document describes several MATLAB scripts that determine if and when a satellite enters the shadow of the Earth or Moon. When a satellite is eclipsed an astrodynamist is interested in such things as the times of shadow entrance and exit, and the Sun-satellite geometry. This type of information is useful for such things as sizing solar panel arrays and batteries, and accessing duty cycle requirements. Shadow knowledge also helps engineers evaluate the thermal control requirements and power management characteristics of satellite subsystems. Shadow characteristics can be used to schedule celestial viewing opportunities from a satellite such as the Hubble Space Telescope and predict ground-based visual observations of large, bright spacecraft such as the International Space Station.

The first MATLAB script determines the eclipse duration and beta angle for satellites in circular orbits. The second and third scripts implement a minimization and root-finding solution to the shadow problem. The fourth script also uses a minimization technique to determine if and when an Earth satellite is eclipsed by the shadow of the Moon. For more realistic shadow predictions, the radius of the Earth is increased by 2% to account for the effect of the atmosphere on the size of the shadow.

These MATLAB applications can be used to determine Earth and lunar shadow conditions of satellites in circular and elliptical Earth orbits. Each script allows the user to save a simple ASCII data file of shadow conditions and also create a graphics display of shadow duration as a function of mission elapsed time.

shadow1.m – shadow conditions of Earth satellites in circular orbits

The approximate eclipse duration of a satellite in a circular orbit that penetrates an Earth shadow represented by a right circular cylinder can be calculated from

$$t_s = \left\{ \cos^{-1}\left(\frac{\sqrt{1-R^2}}{\cos\beta} \right) \right\} \frac{\tau}{\pi}$$

where

R = radius ratio = r_{eq} / r_{sat}

r_{eq} = equatorial radius of the Earth

r_{sat} = geocentric radius of the satellite = $r_{eq} + h_{sat}$

h_{sat} = altitude of the satellite

β = Sun-orbit-plane angle

τ = orbital period of the satellite = $2\pi\sqrt{r_{sat}^3 / \mu}$

μ = gravitational constant of the Earth

This equation also assumes that the Sun does not move during the eclipse. Furthermore, the time units of eclipse duration are the same as the orbital period τ provided the inverse cosine in this equation returns an angle in radians.

The part of the previous equation represented by

$$\cos^{-1}\left(\frac{\sqrt{1-R^2}}{\cos\beta}\right)$$

is one half the true anomaly angle traversed by the satellite during the eclipse. A closer examination of this equation also reveals that satellites at altitudes which satisfy the *radius ratio* inequality given by

$$R > \sin\left|\beta_{\max}\right|$$

will have periods during which they are not eclipsed by the Earth.

The Sun-orbit plane or beta angle is the angle between the geocentric position vector to the Sun and the satellite's orbit plane. It is calculated from

$$\beta = \sin^{-1}\left(\mathbf{r}_{sun} \bullet \mathbf{h}_{sat}\right)$$

where $\mathbf{r}_{sun}$ is the geocentric unit position vector of the Sun and $\mathbf{h}_{sat}$ is the unit angular momentum vector of the satellite's orbit. The unit angular momentum vector is defined by the cross product $\mathbf{r}_{sat} \times \mathbf{v}_{sat}$ where $\mathbf{r}_{sat}$ and $\mathbf{v}_{sat}$ are the unit position and velocity vectors of the satellite, respectively. A positive beta angle indicates that the Sun in on the positive angular momentum side of the orbit plane. The inertial orientation of this vector changes over time as the oblateness of the Earth causes the orbit plane to move. The unit angular momentum vector can also be determined from the satellite's classical orbital elements with

$$\mathbf{h}_{sat} = \begin{Bmatrix} \sin\Omega\sin i \\ -\cos\Omega\sin i \\ \cos i \end{Bmatrix}$$

where i is the satellite's orbital inclination and Ω is the right ascension of the ascending node (RAAN).

The unit position vector of the Sun can be determined with the expression

$$\mathbf{r}_{sun} = \begin{Bmatrix} \cos\delta_{sun}\cos\alpha_{sun} \\ \cos\delta_{sun}\sin\alpha_{sun} \\ \sin\delta_{sun} \end{Bmatrix}$$

where α_{sun} and δ_{sun} are the geocentric, equatorial right ascension and declination of the Sun, respectively.

The beta angle in terms of the satellite's RAAN and orbital inclination is given by

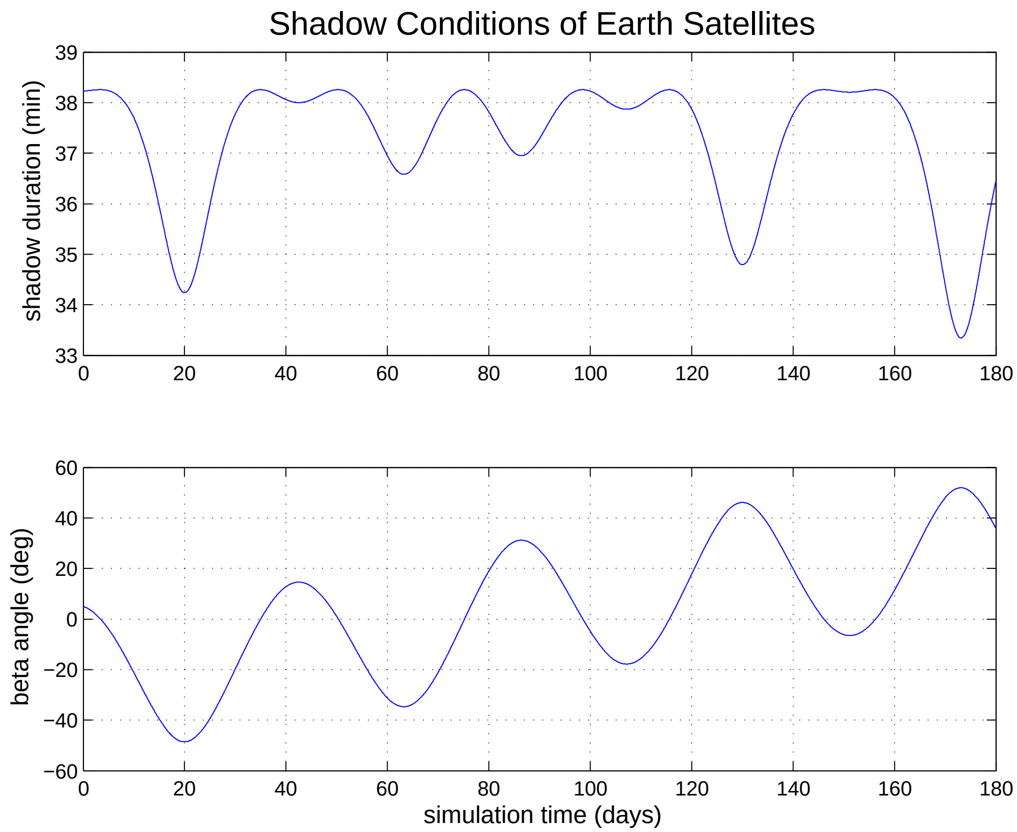
$$\beta = \sin^{-1}\left(\cos\delta_{sun}\sin i\sin\left(\Omega - \alpha_{sun}\right) + \sin\delta_{sun}\cos i\right)$$

Over a period of one year the beta angle varies between

$$-\left(i + \delta_{sun}\right) \le \beta \le +\left(i + \delta_{sun}\right)$$

as the solar declination varies between about plus and minus 23.5°.

An examination of the first equation reveals that the maximum shadow time occurs when $\cos\beta = 1$ which corresponds to $\beta = 0$. This is the instant when the Sun unit position vector r_{sun} lies in the satellite's orbit plane. The minimum shadow time will occur when $\cos\beta$ reaches its minimum value. This happens when β reaches its largest plus and minus values. These relationships are illustrated in the following figure for a satellite at an altitude of 350 kilometers and an inclination of 28.5 degrees. The top graph is a plot of the shadow duration as a function of simulation time and the bottom graph is the beta angle.



The following is a typical user interaction with this program. The software will prompt you for the initial calendar date, the satellite's altitude in kilometers, the initial right ascension of the ascending node in degrees and the orbital inclination of the satellite in degrees. It will also ask you for the total simulation time in days, and the time step to use in minutes. The program will also print shadow statistics consisting of the minimum and maximum values of shadow duration and beta angle, and the average shadow duration. For good statistics, the simulation duration should be long enough such that all relative geometries between the satellite and Sun are simulated.

```
                     program shadow1

 < shadow conditions of Earth satellites in circular orbits >

please input the calendar date
(1 <= month <= 12, 1 <= day <= 31, year = all digits!)
? 1,1,1996

please input the satellite altitude (kilometers)
? 350

please input the orbital inclination (degrees)
(0 <= inclination <= 180)
? 28.5

please input the right ascension of ascending node (degrees)
(0 <= raan <= 360)
? 0

please input the simulation period (days)
? 180

please input the orbit propagation step size (minutes)
? 30
```

The following is an output screen created by the software.

```
shadow conditions of Earth satellites in circular orbits

circular orbit altitude        350.0000  kilometers

orbital inclination             28.5000  degrees

RAAN                             0.0000  degrees

Keplerian period                91.5382  minutes

minimum beta angle             -48.5735  degrees

maximum beta angle              51.9333  degrees

minimum shadow duration         33.3452  minutes

maximum shadow duration         38.2558  minutes

average shadow duration         37.2384  minutes
```

After the program displays this output screen it will prompt the user for a data file and graphics option. If you select the data file option, the program will ask you to input a file name consisting of up to eight characters and a file name extension.

```
would you like to create an ascii data file (y = yes, n = no)
? y

please input the shadow conditions data file name
(be sure to include the file name extension)
? shadow1.txt

would you like to display graphics (y = yes, n = no)
? y
```

The following is the first part of the companion data file. Column one of the output data file is the simulation time in days, column two is the shadow duration in minutes and column three is the beta angle in degrees.

```
 time      duration      beta
(days)     (minutes)   (degrees)

 0.0000     38.2266      4.9879
 0.0208     38.2268      4.9751
 0.0417     38.2269      4.9620
 0.0625     38.2271      4.9487
 0.0833     38.2272      4.9352
 0.1042     38.2274      4.9214
 0.1250     38.2276      4.9075
 0.1458     38.2277      4.8933
 0.1667     38.2279      4.8789
 0.1875     38.2281      4.8642
 0.2083     38.2282      4.8493
 0.2292     38.2284      4.8342
 0.2500     38.2286      4.8189
 0.2708     38.2288      4.8034
 0.2917     38.2289      4.7876
 0.3125     38.2291      4.7716
```

shadow2.m – Kozai orbit propagation

Accurate predictions of shadow conditions for any type of satellite orbit can be determined by using a combination of one-dimensional minimization and root finding. The algorithm used in this MATLAB script searches for minimum values of the angle between the shadow axis and the satellite's position vector as a function of time. If this angle lies within the penumbra angle the algorithm uses Brent's root-finding method to look backward and forward relative to this minimum time to find entrance and exit conditions. This method can also be used to determine entrance and exit relative to the umbra, penumbra and cylindrical shadows.

In this algorithm the objective function we wish to minimize is defined by

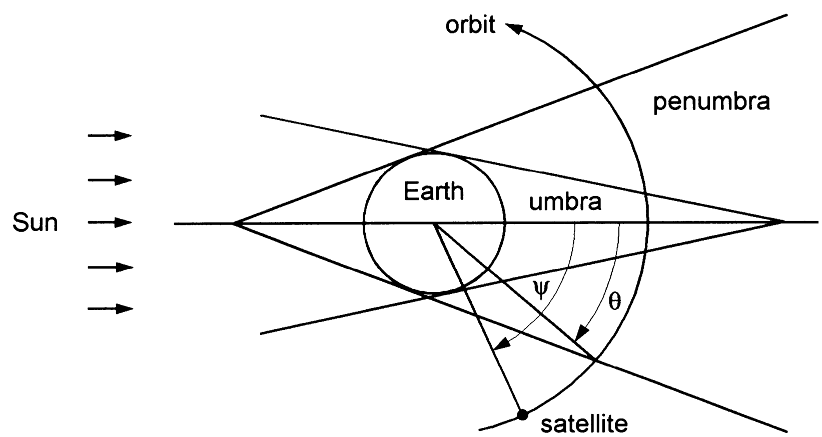
$$f_m = \cos^{-1}\left(-\mathbf{U}_{sun} \bullet \mathbf{U}_{sat}\right)$$

where $\mathbf{U}_{sun}$ and $\mathbf{U}_{sat}$ are the ECI unit pointing vectors of the Sun and satellite, respectively. This scalar value is equal to the angle between the satellite and the anti-Sun vector.

During the root-finding calculations the objective function is given by

$$f_r = \cos^{-1}\left(-\mathbf{U}_{sun} \bullet \mathbf{U}_{sat}\right) - \theta$$

where θ is either the penumbra, umbra or cylinder shadow angle. Before solving for the roots that define shadow entrance and exit, the solution is first bracketed using a geometric acceleration technique. The following diagram illustrates the shadow geometry.



The shadow angles are the angles between the anti-Sun ECI vector and the shadow boundary at the satellite's geocentric distance. The cylindrical shadow angle at the point of shadow entrance or exit is given by:

$$\theta_c = \sin^{-1}\left(\frac{r_{eq}}{r_{sat}}\right)$$

The angle of the umbra portion of the shadow at the satellite's location is determined from

$$\theta_u = \sin^{-1}\left(\frac{d_{sun} - r_{eq}}{r_{sun}}\right) - \theta_c$$

The penumbra shadow angle can be calculated from the following expression:

$$\theta_p = \sin^{-1}\left(\frac{d_{sun} + r_{eq}}{r_{sun}}\right) - \theta_c$$

In these equations, r_{eq} is the equatorial radius of the Earth, d_{sun} is the radius of the Sun, and r_{sat} and r_{sun} are the geocentric distances of the satellite and Sun, respectively.

The phase angle at shadow entrance and exit is the angle between the ECI vectors to the Sun and satellite. The phase angle at the entrance and exit points can be calculated with the equation:

$$\psi = \cos^{-1}\left(\mathbf{U}_{sun} \bullet \mathbf{U}_{sat}\right)$$

where $\mathbf{U}_{sun}$ and $\mathbf{U}_{sat}$ are the ECI unit position vectors of the Sun and satellite, respectively. These unit vectors are evaluated at the points of shadow entrance and exit. The phase angle is an indication of the brightness or illumination of the satellite relative to an Earth observer. The actual brightness of a satellite is a function of its shape, reflective properties, and orientation or attitude in space. For a *spherical* satellite the illuminated fraction can be calculated from

$$I = \frac{1+\cos\psi}{2}$$

The software will ask you for the initial calendar date and Universal Time. It will then ask you to input the classical orbital elements of the satellite and the simulation duration in days. The program will display a shadow type menu that allows you to calculate shadow conditions for the penumbra, umbra or cylindrical portion of the shadow. The program also includes options for screen graphics and a data file. This script uses the Kozai analytic orbit propagation method.

The graphics option will plot the shadow duration as a function of simulation time and the data file option writes a simple ASCII file of elapsed time in days and shadow duration in minutes. Please note that the graphics option plots the shadow duration at the simulation time corresponding to minimum separation angle. Column 1 of the data file is the simulation time in days and column 2 is the shadow duration in minutes. The simulation time corresponds to the time of minimum separation angle.

The following is a typical user interaction with this program.

```
               program shadow2

    < shadow conditions of Earth satellites >

         < Kozai orbit propagation >


   please input the calendar date
   (1 <= month <= 12, 1 <= day <= 31, year = all digits!)
   ? 6,14,1990

   please input the universal time
   (0 <= hours <= 24, 0 <= minutes <= 60, 0 <= seconds <= 60)
   ? 23,0,0

   initial orbital elements

   please input the semimajor axis (kilometers)
   (semimajor axis > 0)
   ? 24450

   please input the orbital eccentricity (non-dimensional)
```

```
(0 <= eccentricity < 1)
? .725

please input the orbital inclination (degrees)
(0 <= inclination <= 180)
? 18

please input the argument of perigee (degrees)
(0 <= argument of perigee <= 360)
? 180

please input the right ascension of the ascending node (degrees)
(0 <= raan <= 360)
? 68

please input the true anomaly (degrees)
(0 <= true anomaly <= 360)
? 0

please input the simulation duration in days
? 100

please select the type of shadow calculation

 <1> penumbra shadow

 <2> umbra shadow

 <3> cylinder shadow

? 2
```

This script will also ask the user if he or she would like to display the numeric results on the screen with the following prompt:

```
would you like screen output (y = yes, n = no)
? y
```

The following is a typical shadow entrance and exit screen display. The software will also calculate and display the event duration.

```
enter shadow conditions

calendar date        14-Jun-1990
universal time       23:00:00

Julian day           2448057.4583

shadow angle            75.1086  degrees

exit shadow conditions

calendar date        14-Jun-1990
universal time       23:15:27
```

```
Julian day           2448057.4691

shadow angle            46.6405  degrees

shadow duration         15.4477  minutes
```

This MATLAB script also allows the user to create an ASCII data file of shadow conditions. The user prompt for this option is

```
would you like to create an ascii data file (y = yes, n = no)
? y

please input the shadow conditions data file name
(be sure to include the file name extension)
? shadow2.txt
```

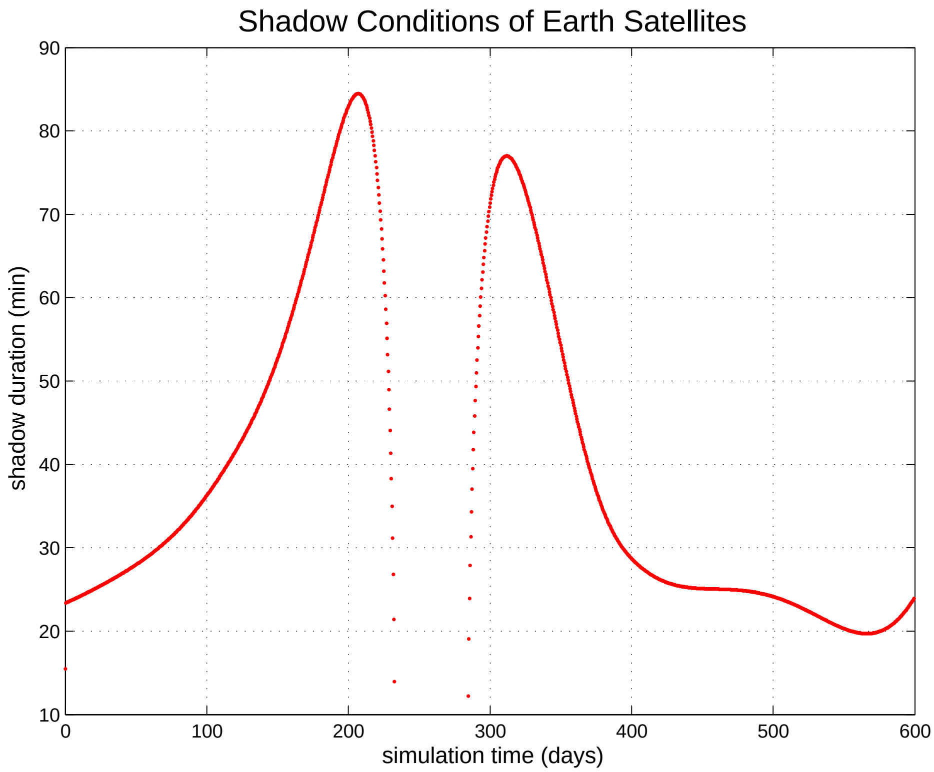
The following is the first part of a typical shadow conditions data file. Column one of the output data file is the shadow entrance time in days, column two is the shadow exit time in days and column three is the shadow duration in minutes.

```
 entrance       exit    duration
  (days)       (days)   (minutes)

   0.0000      0.0045      6.4912
   0.0630      0.0868     34.3908
   0.1453      0.1692     34.4263
   0.2276      0.2515     34.4613
   0.3099      0.3338     34.4958
   0.3922      0.4162     34.5298
   0.4745      0.4985     34.5633
   0.5568      0.5808     34.5964
   0.6391      0.6632     34.6289
   0.7214      0.7455     34.6609
   0.8037      0.8278     34.6925
   0.8861      0.9102     34.7236
   0.9684      0.9925     34.7543
   1.0507      1.0748     34.7844
```

Finally, this script includes an option for creating a graphics display of shadow conditions. The user prompt for this feature is

```
would you like to display graphics (y = yes, n = no)
? y
```

The companion graphics display for this example is shown below.



shadow3.m – numerically integrated orbital motion

This MATLAB script is similar to `shadow2.m` except for the method of orbit propagation. In this version a satellite's equations of motion are numerically integrated during the solution process. This script is "hard-wired" to use a J_2 gravity model. However, it can be easily modified to use orbital equations of motion that include aerodynamic drag or third-body effects for example. This can be accomplished by changing the following line of source code in the `shadfun3.m` support function:

```
ysaved = rkf78('j2eqm', neq, ttmp, x, h, tetol, ytmp);
```

In this code `'j2eqm'` should be replaced by the MATLAB function that computes the new form of the equations of motion.

shadow4.m – lunar eclipse of Earth satellites

This MATLAB application determines lunar eclipse conditions for satellites in elliptic Earth orbits. The user can elect to calculate either penumbra or umbra shadow conditions during a simulation.

Entrance and exit of an Earth satellite relative to the lunar shadow can also be determined using a combination of one-dimensional minimization and root finding. This algorithm searches for minimum values of the angle between the vector defined by the lunar shadow axis and the satellite's *selenocentric* or Moon-centered position vector as a function of time. If this angle lies within the penumbra angle the algorithm uses Brent's root-finding method to look backward and forward in time to find entrance and exit conditions. This method can also be used to determine entrance and exit relative to the umbra shadow of the Moon.

In this algorithm the objective function we wish to minimize is defined by

$$f_m = \cos^{-1}\left(-\mathbf{U}_{sat} \bullet \mathbf{U}_{sun}\right)$$

where $\mathbf{U}_{sat}$ and $\mathbf{U}_{sun}$ are the selenocentric unit pointing vectors of the satellite and Sun, respectively. The calculation of these vectors is given by the expressions

$$\mathbf{U}_{sat} = \frac{\mathbf{r}_{sat} - \mathbf{r}_{moon}}{\left|\mathbf{r}_{sat} - \mathbf{r}_{moon}\right|}$$

and

$$\mathbf{U}_{sun} = \frac{\mathbf{r}_{sun} - \mathbf{r}_{moon}}{\left|\mathbf{r}_{sun} - \mathbf{r}_{moon}\right|}$$

where $\mathbf{r}_{sat}$, $\mathbf{r}_{sun}$ and $\mathbf{r}_{moon}$ are the ECI position vectors of the satellite, Sun and Moon, respectively.

The algorithm propagates the satellite’s orbit forward in time while it brackets minimum values of f_m. After bracketing a minimum, the algorithm calculates the minimum value and checks to see if it is less than the penumbra angle of the Moon's shadow at the time of the minimum. If this single geometric constraint is satisfied, the software performs the root-finding calculations.

During the root-finding calculations the shadow function is given by

$$f_r = \cos^{-1}\left(-\mathbf{U}_{sat} \bullet \mathbf{U}_{sun}\right) - \theta$$

where θ is either the penumbra or umbra angle of the lunar shadow. The numerical method also brackets roots of this function before it solves this equation.

The angle of the penumbra portion of the lunar shadow at the satellite's selenocentric distance is given by

$$\theta_p = \sin^{-1}\left(\frac{d_{moon}}{r_{moon-to-sat}}\right) + \sin^{-1}\left(\frac{d_{sun} + d_{moon}}{r_{moon-to-sun}}\right)$$

where d_{moon} and d_{sun} are the radius of the Moon and Sun, respectively. In this equation $r_{moon-to-sat}$ is the distance from the Moon to the satellite and $r_{moon-to-sun}$ is the distance from the Moon to the Sun.

The length of the umbra shadow cast by the Moon can be calculated from

$$l_u = \frac{d_{moon} r_{moon-to-sat}}{d_{sun} - d_{moon}}$$

The umbra angle of the Moon's shadow at the satellite's orbital distance is determined from

$$\theta_u = \sin^{-1}\left(\frac{d \sin\alpha}{r_{moon-to-sat}}\right)$$

where $d = r_{moon-to-sat} - l_u$ if $r_{moon-to-sat} > l_u$ or $d = l_u - r_{moon2sat}$ otherwise. In this equation, $\sin\alpha = \left(d_{sun} - d_{moon}\right) / r_{moon-to-sun}$.

The following is a typical user interaction with this program. This simulation predicts the lunar eclipse durations for a typical geosynchronous satellite.

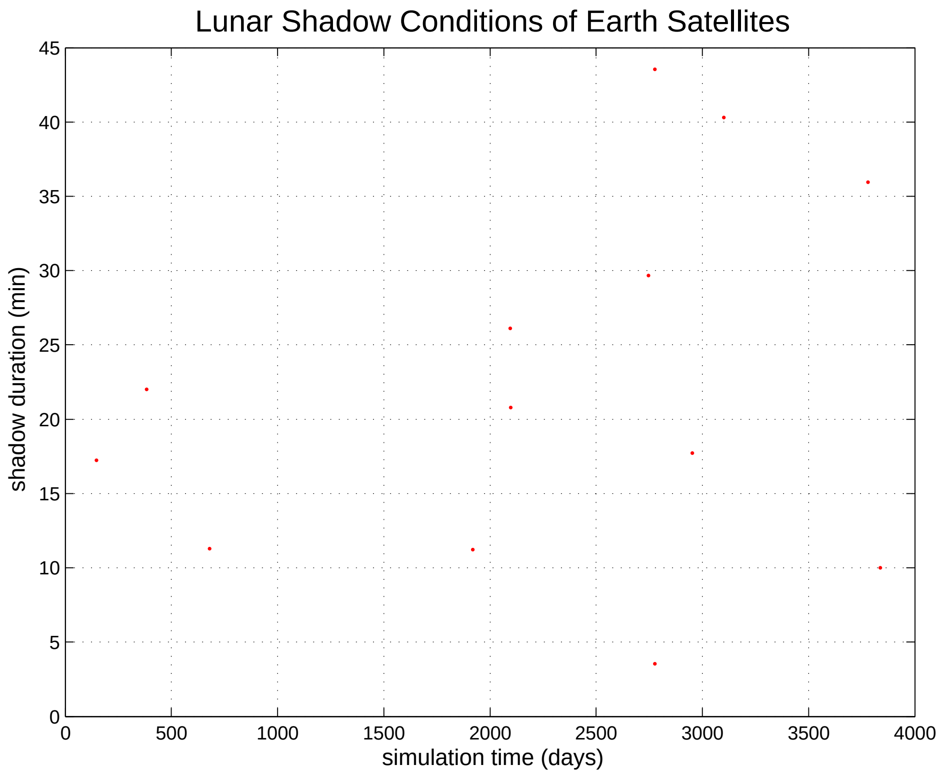
```
                    program shadow4

    < lunar shadow conditions of Earth satellites >

              < Kozai orbit propagation >


   please input the calendar date
   (1 <= month <= 12, 1 <= day <= 31, year = all digits!)
   ? 7,12,1991

   please input the universal time
   (0 <= hours <= 24, 0 <= minutes <= 60, 0 <= seconds <= 60)
   ? 5,0,0

   initial orbital elements

   please input the semimajor axis (kilometers)
   (semimajor axis > 0)
   ? 42164.5

   please input the orbital eccentricity (non-dimensional)
   (0 <= eccentricity < 1)
   ? 0

   please input the orbital inclination (degrees)
   (0 <= inclination <= 180)
   ? 0

   please input the true anomaly (degrees)
   (0 <= true anomaly <= 360)
   ? 291.45

   please select the type of shadow calculation
```

```
 <1> penumbra shadow

 <2> umbra shadow

? 2

please input the simulation duration in days
? 500
```

The following is a typical shadow entrance and exit screen display.

```
enter shadow conditions

calendar date        06-Dec-1991
universal time       04:22:23

Julian day           2448596.6822

shadow angle             0.2694  degrees


exit shadow conditions

calendar date        06-Dec-1991
universal time       04:39:37

Julian day           2448596.6942

shadow angle             0.2694  degrees

shadow duration         17.2269  minutes
```

This MATLAB script includes options for creating a data file and graphics display of visibility conditions. The following plot shows the shadow durations for this simulation for a period of 4000 days. It illustrates that lunar eclipses of Earth satellites are rare.


Lunar Shadow Conditions of Earth Satellites
shadow duration (min)
simulation time (days)
0
5
10
15
20
25
30
35
40
45
0
500
1000
1500
2000
2500
3000
3500
4000