Danby's Solution of Kepler's Equation

This program (`demokep1`) demonstrates how to interact with the ***Numerit*** function `kepler1` which solves the elliptic and hyperbolic form of Kepler's equation using a numerical method devised by Professor J.M.A. (Tony) Danby at North Carolina State University. Additional information about this algorithm can be found in the technical papers "The Solution of Kepler's Equation", *Celestial Mechanics*, **31** (1983) 95-107, 317-328 and **40** (1987) 303-312.

The *elliptic* form of Kepler's equation is given by the expression

$$M = E - e\sin E \tag{1}$$

where M is the mean anomaly, E is the eccentric anomaly and e is the orbital eccentricity.

The eccentric anomaly *initial guess* for Danby's method is

$$E_0 = M + 0.85\, sign(\sin M)\, e \tag{2}$$

The *fundamental equation* we want to solve is

$$f(E) = E - e\sin E - M = 0 \tag{3}$$

which has the first three derivatives given by

$$\begin{aligned} f'(E) &= 1 - e\cos E \\ f''(E) &= e\sin E \\ f'''(E) &= e\cos E \end{aligned} \tag{4}$$

The iteration for an *updated* eccentric anomaly E_{n+1} based on a current value E_n is given by the next four equations:

$$\begin{aligned} \Delta(E_n) &= -\frac{f}{f'} \\ \Delta^*(E_n) &= -\frac{f}{f' + \frac{1}{2}\Delta f''} \\ \Delta_n(E_n) &= -\frac{f}{f' + \frac{1}{2}\Delta f'' + \frac{1}{6}\Delta^{*2} f'''} \\ E_{n+1} &= E_n + \Delta_n \end{aligned} \tag{5}$$

This algorithm provides *quartic* convergence of Kepler's equation. The process summarized by Eq. 5 is repeated until the following convergence test involving the fundamental equation is satisfied:

$$\left| f(E) \right| \le e \tag{6}$$

where e is the convergence tolerance. This tolerance is "hardwired" in the software to the value `ktol = 1.0e-10`.

Finally, the true anomaly can be calculated with the following two equations

$$\begin{aligned} \sin n &= \sqrt{1-e^2}\,\sin E \\ \cos n &= \cos E - e \end{aligned} \tag{7}$$

and the four quadrant inverse tangent given by

$$n = \arctan\left(\sin n,\ \cos n\right) \tag{8}$$

If the orbit is hyperbolic the initial guess is

$$H_0 = \log\left(\frac{2M}{e} + 1.8\right) \tag{9}$$

where H_0 is the hyperbolic anomaly.

The fundamental equation and first three derivatives for this case are as follows:

$$\begin{aligned} f(H) &= e\sinh H - H - M \\ f'(H) &= e\cosh H - 1 \\ f''(H) &= e\sinh H \\ f'''(H) &= e\cosh H \end{aligned} \tag{10}$$

Otherwise, the iteration loop which calculates Δ, Δ^* and so forth is the same. The true anomaly for hyperbolic orbits is determined with the next two equations and a four quadrant inverse tangent function as before.

$$\begin{aligned} \sin n &= \sqrt{e^2-1}\,\sinh H \\ \cos n &= e - \cosh H \end{aligned} \tag{11}$$

The syntax for this ***Numerit*** function is as follows:

```
function kepler1 (manom, ecc, eanom, tanom)

` solve Kepler's equation for circular,
` elliptic and hyperbolic orbits

` Danby's method

` input

`  manom = mean anomaly (radians)
`  ecc   = orbital eccentricity (non-dimensional)

` output

`  eanom = eccentric anomaly (radians)
`  tanom = true anomaly (radians)
```

The program will interactively prompt you for the orbital eccentricity. It will then calculate and plot arrays containing the number of algorithm iterations and the algorithm *absolute* error (Equation 3).

The following are plots of the number of algorithm iterations and the absolute error for an orbital eccentricity of 0.735. This is the eccentricity of a typical Molniya orbit. For this example Danby's method requires at most 3 iterations and the absolute value of the algorithm error is always less than 10^{-12}.

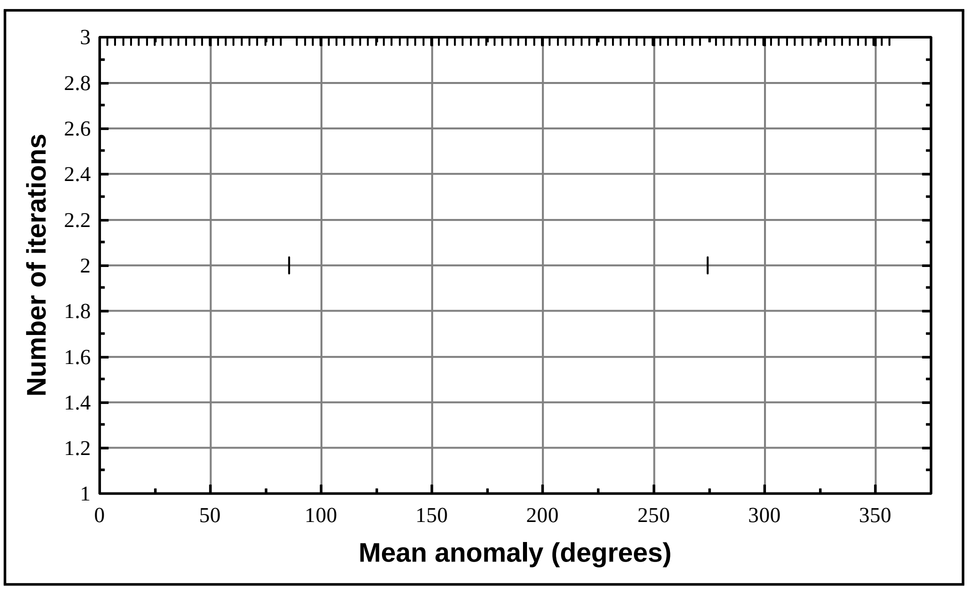

Figure 1. Algorithm iterations

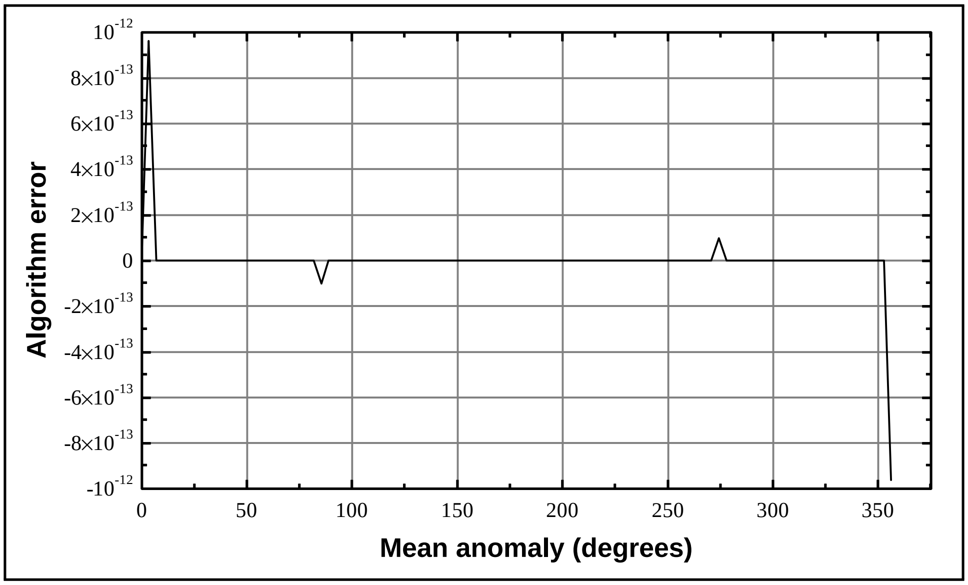

Figure 2. Algorithm error