Kepler's Equation for Parabolic and Near-parabolic Heliocentric Orbits

This program (`demokep2`) demonstrates how to interact with the ***Numerit*** function `kepler2` which solves Kepler's equation for heliocentric parabolic and near-parabolic orbits. It is based on the numerical method described in Chapter 4 of *Astronomy on the Personal Computer* by Oliver Montenbruck and Thomas Pfleger. This algorithm uses a modified form of *Barker's equation* and *Stumpff functions* to solve this problem.

The form of Kepler's equation solved by this function is

$$E(t) - e\sin E(t) = \sqrt{\frac{m}{a^3}}\,(t - t_0) \tag{1}$$

where

$E =$ eccentric anomaly
$e =$ orbital eccentricity
$m =$ gravitational constant of the Sun
$a =$ semimajor axis
$t =$ time
$t_0 =$ time of perihelion passage

The relationship between semimajor axis a and perihelion radius q is as follows

$$a = \frac{q}{1-e} \tag{2}$$

By introducing the variable

$$U = \sqrt{\frac{3e\,c_3(E)}{1-e}}\,E \tag{3}$$

Kepler's equation is now given by

$$U + \frac{1}{3}U^3 = \sqrt{6\,e\,c_3(E)}\,\sqrt{\frac{m}{2q^3}}\,(t - t_0) \tag{4}$$

where $c_3(E) = (E - e\sin E)/E^3$. The `kepler2` function iteratively solves for U. The heliocentric distance is determined from the expression

$$r = q\left(1 + \left[\frac{2\,c_2}{6\,c_3}\right]U^2\right) \tag{5}$$

The true anomaly is determined from the x and y components of the heliocentric position vector as follows

$$n = \operatorname{atan}(y, x) \tag{6}$$

where

$$x = q\left(1 - \left[\frac{2c_2}{6ec_3}\right]U^2\right)$$

$$y = 2q\sqrt{\frac{1+e}{2e}\left[\frac{1}{6c_3}\right]}c_1 U$$

The true anomaly can also be determined from

$$\tan\left(\frac{n}{2}\right) = \left[\sqrt{\frac{1+e}{3ec_3}}\frac{c_2}{c_1}\right]U \tag{7}$$

The *c* functions used in these equations are called Stumpff functions. They are named after the German astronomer Karl Stumpff and defined by the series

$$c_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{(2k+n)!} \qquad k = 0, 1, 2, \ldots \tag{8}$$

For x real and $x \neq 0$, the first few terms are given by the following expressions

$$\begin{aligned} c_0(x^2) &= \cos x & c_0(-x^2) &= \cosh x \\ c_1(x^2) &= \frac{\sin x}{x} & c_1(-x^2) &= \frac{\sinh x}{x} \\ c_2(x^2) &= \frac{1-\cos x}{x} & c_2(-x^2) &= \frac{\cosh x - 1}{x} \end{aligned} \tag{9}$$

The Stumpff functions also satisfy the recursion relationship defined by

$$x\,c_{k+2}(x) = \frac{1}{k!} - c_k(x) \qquad k = 0, 1, 2, \ldots \tag{10}$$

For $x = 0$,

$$c_n(x) = \frac{1}{n!} \tag{11}$$

It is most efficient to compute c_2 and c_3 by series, and then compute c_0 and c_1 by recursion according to the following:

$$c_0(x) = 1 - x c_2$$
$$c_1(x) = 1 - x c_3 \qquad (12)$$

The program will prompt you for the perihelion radius in Astronomical Units, the time relative to perihelion passage in days and the orbital eccentricity. Please note that times prior to perihelion passage should be input as negative numbers. Times after perihelion passage are positive numbers.

The following is a typical draft output created with this program.

```
perihelion radius (AU)             =  0.6

orbital eccentricity               =  0.9985000000000001

time since perihelion passage (days) =  -60

heliocentric distance (AU)         =  1.341035285746319

true anomaly (degrees)             =  263.9152690390323
```