State Vectors and Orbital Elements

This program (`demosvoe`) demonstrates the relationship between *classical orbital elements* and the *state vector* of a space object in an elliptical orbit. The state vector consists of the inertial position and velocity vectors of the object relative to the central body. The following is a brief description of the classical orbital elements.

The *semimajor axis* defines the size of the orbit and the *orbital eccentricity* defines the shape of the orbit. The angular orbital elements are defined with respect to a fundamental x-axis, the vernal equinox, and a fundamental plane, the equator. The z-axis of this system is collinear with the spin axis of the Earth, and the y-axis completes a right-handed coordinate system.

The *orbital inclination* is the angle between the equatorial plane and the orbit plane. Satellite orbits with inclinations between 0 and 90 degrees are called *direct* orbits and satellites with inclinations greater than 90 and less than 180 degrees are called *retrograde* orbits. The *right ascension of the ascending node* (RAAN) is the angle measured from the x-axis (vernal equinox) eastward along the equator to the ascending node. The *argument of perigee* is the angle from the ascending node, measured along the orbit plane in the direction of increasing true anomaly, to the argument of perigee. The *true anomaly* is the angle from the argument of perigee, measured along the orbit plane in the direction of motion, to the satellite's location. Finally, the *argument of latitude* is the angle from the ascending node, measured in the orbit plane, to the satellite's location in the orbit. The argument of latitude is equal to $u = n + w$.

The *orbital eccentricity* is an indication of the type of orbit. For values of $0 \le e < 1$, the orbit is circular or elliptic. The orbit is parabolic when $e = 1$ and the orbit is hyperbolic if the condition $e > 1$ is true.

The semimajor axis a is calculated using the following expression:

$$a = \frac{1}{\frac{2}{r} - \frac{v^2}{m}} \tag{1}$$

where $r = |\mathbf{r}| = \sqrt{r_x^2 + r_y^2 + r_z^2}$ is the scalar position and $v = |\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$ is the scalar velocity or speed of the space object.

The angular orbital elements are calculated from the *equinoctial* orbital elements h, k, p and q which are in turn calculated from the rectangular components of the body-centered inertial position and velocity vectors. The equinoctial orbital elements are an alternative set of *non-singular* elements which avoid computational problems when working with orbits with small eccentricity or inclination.

The mathematical relationship between equinoctial and classical orbital elements is given by the following expressions:

$$\begin{aligned} a &= a \\ h &= e\sin(w+\Omega) \\ k &= e\cos(w+\Omega) \\ l &= M+w+\Omega \\ p &= \tan(i/2)\sin\Omega \\ q &= \tan(i/2)\cos\Omega \end{aligned} \qquad (2)$$

In the fourth equation M is the mean anomaly and l is called the mean longitude.

The scalar orbital eccentricity e is determined from h and k as follows:

$$e = \sqrt{h^2+k^2} \qquad (3)$$

The orbital inclination i is determined from p and q using the following expression

$$i = 2\arctan\left(\sqrt{p^2+q^2}\right) \qquad (4)$$

For values of inclination greater than a small value e, the right ascension of the ascending node Ω is given by

$$\Omega = \arctan(p, q) \qquad (5)$$

Otherwise, the orbit is equatorial and there is no RAAN.

If the value of orbital eccentricity is greater than e, the argument of perigee w is determined from

$$w = \arctan(h, k) - \Omega \qquad (6)$$

Otherwise, the orbit is circular and there is no argument of perigee. In the ***Numerit*** code for these calculations, $e = 10^{-8}$.

Finally, the true anomaly n is found from the expression

$$n = l - \Omega - w \qquad (7)$$

In these equations, all two argument inverse tangent calculations use a four quadrant ***Numerit*** function called `atan3` to determine the correct quadrant for the angle. Angular orbital elements which can range from 0 to 360 degrees are also processed with a modulo 2*p* function named `modulo`.

The body-centered, inertial rectangular components of the radius vector can be determined from the classical orbital elements as follows:

$$r_x = p\left[\cos\Omega\cos(w+n) - \sin\Omega\cos i\sin(w+n)\right]$$

$$r_y = p\left[\sin\Omega\cos(w+n) + \cos\Omega\cos i\sin(w+n)\right] \qquad (8)$$

$$r_z = p\sin i\sin(w+n)$$

The rectangular components of the velocity vector are given by

$$v_x = -\sqrt{\frac{m}{p}}\left[\cos\Omega\left\{\sin(w+n) + e\sin w\right\} + \sin\Omega\cos i\left\{\cos(w+n) + e\cos w\right\}\right]$$

$$v_y = -\sqrt{\frac{m}{p}}\left[\sin\Omega\left\{\sin(w+n) + e\sin w\right\} - \cos\Omega\cos i\left\{\cos(w+n) + e\cos w\right\}\right] \qquad (9)$$

$$v_z = -\sqrt{\frac{m}{p}}\left[\sin i\left\{\cos(w+n) + e\cos w\right\}\right]$$

In these equations p is called the *semiparameter* of the orbit and is calculated from $p = a(1-e^2)$. m is the gravitational constant of the *primary* or central body.

The syntax of the function which converts the state vector to classical orbital elements is

```
function eci2orb (mu, r, v, oev)

` convert eci state vector to six
` classical orbital elements via
` equinoctial elements

` input

`  mu = gravitational constant (km^3/sec^2)
`  r  = eci position vector (kilometers)
`  v  = eci velocity vector (kilometers/second)

` output

`  oev[1]  = semimajor axis (kilometers)
`  oev[2]  = orbital eccentricity (non-dimensional)
`            (0 <= eccentricity < 1)
`  oev[3]  = orbital inclination (radians)
`            (0 <= inclination <= pi)
`  oev[4]  = argument of perigee (radians)
`            (0 <= argument of perigee <= 2 pi)
`  oev[5]  = right ascension of ascending node (radians)
`            (0 <= raan <= 2 pi)
`  oev[6]  = true anomaly (radians)
`              (0 <= true anomaly <= 2 pi)
```

The syntax of the ***Numerit*** function which converts the classical orbital elements to a state vector is as follows:

```
function orb2eci(mu, oev, r, v)

` convert classical orbital elements to eci state vector

` input

`  mu     = gravitational constant (km^3/sec^2)
`  oev[1] = semimajor axis (kilometers)
`  oev[2] = orbital eccentricity (non-dimensional)
`           (0 <= eccentricity < 1)
`  oev[3] = orbital inclination (radians)
`           (0 <= inclination <= pi)
`  oev[4] = argument of perigee (radians)
`           (0 <= argument of perigee <= 2 pi)
`  oev[5] = right ascension of ascending node (radians)
`           (0 <= raan <= 2 pi)
`  oev[6] = true anomaly (radians)
`           (0 <= true anomaly <= 2 pi)

` output

`  r = eci position vector (kilometers)
`  v = eci velocity vector (kilometers/second)
```

The following is a typical draft output created with this program.

```
position vector (kilometers)

rx =  7475.226183658003
ry =  1103.012821501304
rz =  2150.118648247414

velocity vector (kilometers/second)

vx =  -0.04900375055806951
vy =  6.629471263012779
vz =  -2.774486590207703

orbital elements

semimajor axis                     (kilometers) =  7999.999999999996
orbital eccentricity          (non-dimensional) =  0.0249999999999999
orbital inclination                   (degrees) =  28.5
argument of perigee                   (degrees) =  99.99999999999959
right ascension of the ascending node (degrees) =  220
true anomaly                          (degrees) =  45.00000000000041
```