Frozen Orbit Design

A frozen orbit is characterized by no long-term changes in orbital eccentricity and argument of perigee. This type of orbit maintains almost constant altitude over any particular point on a planet's surface. The design of frozen orbits involves selecting the correct value of eccentricity and argument, for a given semimajor axis and orbital inclination, which satisfies the following system of nonlinear perturbation equations:

$$\frac{de}{dt} = \frac{3}{2}\frac{J_3 r_{eq}^3}{p^3}\left(1-e^2\right) n \sin i \cos w \left(\frac{5}{4}\sin^2 i - 1\right) = 0$$

$$\frac{dw}{dt} = \frac{3}{2}\frac{J_2 r_{eq}^2}{p^2} n \left(2 - \frac{5}{2}\sin^2 i\right) - \frac{3}{2}\frac{J_3 r_{eq}^3 \sin w}{p^3 e \sin i} n \left\{ \begin{gathered} \left(\frac{5}{4}\sin^2 i - 1\right)\sin^2 i \\ + e^2\left(1 - \frac{35}{4}\sin^2 i \cos^2 i\right) \end{gathered} \right\} = 0 \qquad (1)$$

where

$a =$ semimajor axis
$e =$ orbital eccentricity
$i =$ orbital inclination
$w =$ argument of perigee
$p = a\left(1-e^2\right) =$ semiparameter
$r_{eq} =$ Earth equatorial radius
$m =$ Earth gravitational constant
$n = \sqrt{m/a^3} =$ mean motion
$J_2 =$ second gravity coefficient
$J_3 =$ third gravity coefficient

By solving this nonlinear system of equations, the perturbing effect of the J_2 even zonal gravity harmonic on eccentricity and argument of perigee are balanced by the J_3 odd zonal gravity harmonic. For argument of perigee values equal to 90° and 270°, the eccentricity perturbation vanishes.

This ***Numerit*** program (`frozen1`) determines the mean orbital eccentricity required for a frozen orbit. The user inputs the mean semimajor axis and orbital inclination and the program calculates the eccentricity using Brent's method to solve the single constraining nonlinear equation. The software also calculates and displays the real roots of the frozen orbit eccentricity cubic equation given by:

$$a_1 e^3 + a_2 e^2 + a_3 e + a_4 = 0 \qquad (2)$$

The derivation of this equation can be found in the technical report, "Frozen Orbits in the J2 + J3 Problem", by Krystyna Kiedron and Richard Cook, AAS 91-426, AAS/AIAA Astrodynamics Specialist Conference, Durango, Colorado, August 19-22, 1991.

The coefficients of this cubic equation are given by

$$
\begin{aligned}
a_1 &= -\frac{3}{4} n \left(\frac{r_{eq}}{a} \right)^2 J_2 \sin i \left(1 - 5\cos^2 i \right) \\
a_2 &= \frac{3}{2} n \left(\frac{r_{eq}}{a} \right)^3 J_3 \left(1 - \frac{35}{4} \sin^2 i \cos^2 i \right) \\
a_3 &= -a_1 \\
a_4 &= \frac{3}{2} n \left(\frac{r_{eq}}{a} \right)^3 J_3 \sin^2 i \left(\frac{5}{4} \sin^2 i - 1 \right)
\end{aligned}
\tag{3}
$$

The following is a typical draft output created with this program:

```
program frozen1

< orbital eccentricity of frozen orbits >

mean orbital elements

semimajor axis         8000  kilometers
eccentricity           0.0006594137728
inclination            45  degrees
argument of perigee    90  degrees

real roots of the frozen eccentricity cubic equation

0.0006594137728
-1.002419172
0.9975834848
```

From this example we can see that the mean orbital eccentricity computed by Brent's method and the cubic equation agree quite well.