Relative Motion Between Two Satellites in Circular Earth Orbits

This ***Numerit*** program (`grmotion`) can be used to graphically display the relative two-dimensional motion between two satellites in circular Earth orbits. The software provides the following user options:

- user input of initial conditions (position and ΔV)
- calculate and display the characteristics of a synchronous orbit
- calculate and display the characteristics of a rendezvous orbit

In the following discussion the passive satellite is called the *target* and the active or maneuvering satellite is called the *chaser*. The state vector of the chaser satellite is defined with respect to a local vertical-local horizontal (LVLH) coordinate system centered at the target satellite. The positive x-axis of this system is in the direction of orbital motion, the positive y-axis is in the radial direction away from the geocenter and the z-axis completes this right-handed coordinate system.

The relationship between the position and velocity vectors of a chaser vehicle at any time t to the initial position and velocity vectors at time t_0 is given by the following *state transition matrix for unperturbed relative motion*:

$$\begin{Bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{Bmatrix} = \begin{bmatrix} 1 & -6(\omega t - \sin\omega t) & 0 & -3t + \dfrac{4}{\omega}\sin\omega t & -\dfrac{2}{\omega}(1-\cos\omega t) & 0 \\ 0 & 4 - 3\cos\omega t & 0 & \dfrac{2}{\omega}(1+\cos\omega t) & \dfrac{1}{\omega}\sin\omega t & 0 \\ 0 & 0 & \cos\omega t & 0 & 0 & \dfrac{1}{\omega}\sin\omega t \\ 0 & 6\omega(1-\cos\omega t) & 0 & -3 + 4\cos\omega t & 2\sin\omega t & 0 \\ 0 & 3\omega\sin\omega t & 0 & -2\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & -\omega\sin\omega t & 0 & 0 & \cos\omega t \end{bmatrix} \begin{Bmatrix} x_0 \\ y_0 \\ z_0 \\ \dot{x}_0 \\ \dot{y}_0 \\ \dot{z}_0 \end{Bmatrix} \tag{1}$$

where

$$\omega = \sqrt{\frac{\mu}{(r/r_e)^3}} = \text{ orbital rate}$$

$r =$ radius of circular orbit
$r_e =$ radius of the Earth
$\mu =$ gravitational constant of the Earth

The x, y and z position vector components of the chaser vehicle as a function of time are given by

$$x(t) = \left(x_0 - \frac{2\dot{y}_0}{\omega}\right) + \left(-3\dot{x}_0 - 6\omega y_0\right)t + \left(2\frac{\dot{y}_0}{\omega}\right)\cos\omega t + \left(4\frac{\dot{x}_0}{\omega} + 6y_0\right)\sin\omega t$$

$$y(t) = \left(2\frac{\dot{x}_0}{\omega} + 4y_0\right) + \left(-2\frac{\dot{x}_0}{\omega} - 3y_0\right)\cos\omega t + \left(\frac{\dot{y}_0}{\omega}\right)\sin\omega t \qquad (2)$$

$$z(t) = z_0\cos\omega t + \frac{\dot{z}_0}{\omega}\sin\omega t$$

The relative motion trajectory of the chaser spacecraft is a "drifting" ellipse with its center located at (c,d) where

$$c = x_0 - 2\frac{\dot{y}_0}{\omega} - \left(3\dot{x}_0 + 6\omega y_0\right)t$$
$$d = \frac{2\dot{x}_0}{\omega} + 4y_0 \qquad (3)$$

The semimajor axis of this dynamic ellipse is given by

$$a = 2\left\{\left(-2\frac{\dot{x}_0}{\omega} - 3y_0\right)^2 + \left(\frac{\dot{y}_0}{\omega}\right)^2\right\} \qquad (4)$$

Synchronous orbit

The initial velocity components required for a chaser satellite to be synchronous or "co-orbital" with a *target* satellite in a user-defined circular orbit are given by

$$\dot{x}_{0s} = -2\omega y_0$$
$$\dot{y}_{0s} = 0 \qquad (5)$$

The initial velocity components for any initial x_0 and y_0 are given by

$$\dot{x}_0 = -\frac{3}{2}\omega y_0$$

$$\dot{y}_0 = \frac{\frac{3}{2}\omega x_0 y_0}{\frac{r_e}{r} + y_0} \qquad (6)$$

Therefore, the components of the initial velocity increment for a synchronous orbit are given by

$$\Delta V_x = \dot{x}_{0s} - \dot{x}_0$$
$$\Delta V_y = \dot{y}_{0s} - \dot{y}_0 \tag{7}$$

The relative motion trajectory is an ellipse with its center located at (c,0) where

$$c = x_0 - 2\frac{\dot{y}_0}{\omega} \tag{8}$$

The semimajor axis of this ellipse is given by

$$a = 2\sqrt{y_0^2 + \left(\frac{\dot{y}_0}{\omega}\right)^2} \tag{9}$$

and the semiminor axis is equal to $a/2$.

Rendezvous orbit

Orbital rendezvous is the process of bringing a chaser vehicle from some initial location to a final location with zero relative velocity in a specified transfer time. This type of orbit transfer involves an initial maneuver that starts the transfer and a second maneuver that stops the chaser spacecraft at the final location. This program option calculates the magnitude and direction of these two *impulsive* maneuvers and graphically displays the transfer trajectory.

The initial *terminal phase initiation* velocity components of the rendezvous orbit are given by

$$\dot{x}_{0_{TPI}} = \frac{14y_0\left(1 - \cos\omega t_r\right) - \left(6y_0\,\omega t_r - x_0\right)\sin\omega t_r}{t_r\left[3\sin\omega t_r - \dfrac{8}{\omega t_r}\left(1 - \cos\omega t_r\right)\right]}$$
$$\dot{y}_{0_{TPI}} = \frac{-y_0\left(3\omega t_r\cos\omega t_r - 4\sin\omega t_r\right) - 2x_0\left(1 - \cos\omega t_r\right)}{t_r\left[3\sin\omega t_r - \dfrac{8}{\omega t_r}\left(1 - \cos\omega t_r\right)\right]} \tag{10}$$

where t_r is the user-specified time required to perform the orbit transfer.

The components of the terminal phase initiation ΔV required to initiate the rendezvous maneuver are determined from the following equations

$$\Delta V_x = \dot{x}_{0r} - \dot{x}_0$$
$$\Delta V_y = \dot{y}_{0r} - \dot{y}_0 \tag{11}$$

The components of the ΔV required to <u>brake</u> the chaser vehicle at the target are given by

$$\dot{x}_b(t_r) = (-3\dot{x}_{tpi} - 6\omega y_0) + (-2\dot{y}_{tpi})\sin\omega t_r + (4\dot{x}_{tpi} + 6\omega y_0)\cos\omega t_r$$
$$\dot{y}_b(t_r) = (2\dot{x}_{tpi} + 3\omega y_0)\sin\omega t_r + (\dot{y}_{tpi})\cos\omega t_r \qquad (12)$$

where $\dot{x}_{tpi}$ and $\dot{y}_{tpi}$ are the velocity components of the chaser vehicle when it reaches the location of the target vehicle.

The following is a typical draft report created with this software. It summarizes the maneuver characteristics for a rendezvous trajectory.

```
program grmotion

< graphics display of relative motion >

rendezvous orbit

target altitude            300 kilometers

chaser x distance           50 kilometers
chaser y distance         -100 kilometers

time to rendezvous         120 minutes

vx prior to tpi       173.5309 meters/second
vy prior to tpi      -1.318997 meters/second

tpi delta-vx          94.67525 meters/second
tpi delta-vy         -179.0341 meters/second
tpi delta-v           202.5256 meters/second

braking delta-vx       36.8316 meters/second
braking delta-vy      250.9075 meters/second
braking delta-v       253.5964 meters/second

total delta-v          456.122 meters/second
```

The following is a plot of this rendezvous trajectory. The scales have been manually manipulated to create a picture which is to scale. The target vehicle is indicated by the blue cross symbol and the initial location of the chaser vehicle is indicated by the red diamond symbol. The plot step size for this example was one minute. The orbital motion of the target satellite is to the right.

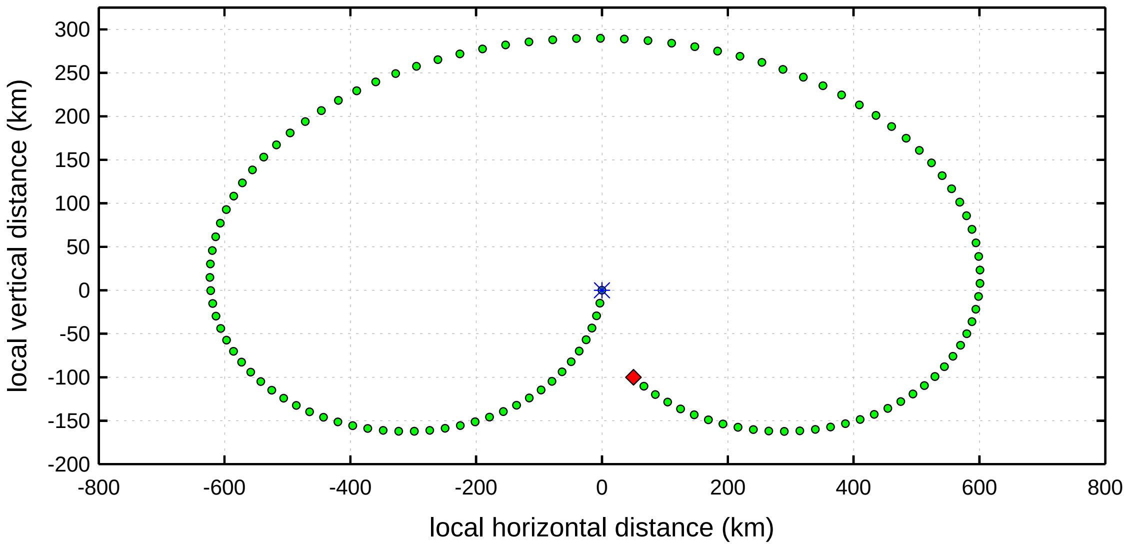

Figure 1. Relative Motion Between Two Earth Satellites