The Hohmann Orbit Transfer

The coplanar circular orbit-to-circular orbit transfer was discovered by the German engineer Walter Hohmann in 1925. The transfer consists of a velocity impulse on an initial circular orbit, in the direction of motion and collinear with the velocity vector, which propels the space vehicle into an elliptical transfer orbit. At a transfer angle of 180 degrees from the first impulse, a second velocity impulse or ΔV, also collinear and in the direction of motion, places the vehicle into a final circular orbit at the desired final altitude. The impulsive ΔV assumption means that the velocity, but not the position, of the vehicle is changed instantaneously. This is equivalent to a rocket engine with infinite thrust magnitude. Therefore, the Hohmann formulation is the ideal and minimum energy solution to this type of orbit transfer problem.

Coplanar Analysis

For the coplanar Hohmann transfer both velocity impulses are confined to the orbital planes of the initial and final orbits. The first impulse creates an elliptical transfer orbit with a perigee altitude equal to the altitude of the initial circular orbit and an apogee altitude equal to the altitude of the final orbit. The second impulse circularizes the transfer orbit at apogee. Both impulses are posigrade which means that they are in the direction of orbital motion.

We begin by defining three *normalized* radii as follows:

$$\begin{aligned} R_1 &= \sqrt{2\frac{r_f}{r_i + r_f}} \\ R_2 &= \sqrt{\frac{r_i}{r_f}} \\ R_3 &= \sqrt{2\frac{r_i}{r_i + r_f}} \end{aligned} \tag{1}$$

where r_i is the geocentric radius of the initial circular park orbit and r_f is the radius of the final circular mission orbit. The relationship between radius and initial orbit altitude h_i and the final orbit altitude h_f is as follows:

$$\begin{aligned} r_i &= h_i + r_{eq} \\ r_f &= h_f + r_{eq} \end{aligned} \tag{2}$$

where r_{eq} is the radius of the Earth.

The magnitude of the first impulse is

$$\Delta V_1 = V_{lc}\sqrt{1 + R_1^2 - 2R_1} \tag{3}$$

and is simply the difference between the speed on the initial orbit and the perigee speed of the transfer orbit. The scalar magnitude of the second impulse is

$$\Delta V_2 = V_{lc}\sqrt{R_2^2 + R_1^2 R_3^2 - 2R_2^2 R_3} \tag{4}$$

which is the difference between the speed on the final orbit and the apogee speed of the transfer ellipse. In both of these ΔV equations V_{lc} is called the *local circular velocity*. It can be determined from $V_{lc_1} = \sqrt{\mu / r_i}$ which represents the scalar speed in the initial orbit. In this equation, μ is the gravitational constant of the central body.

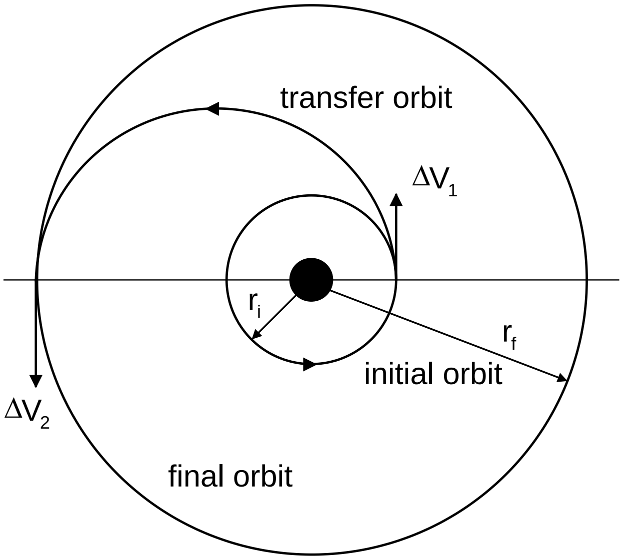
The transfer time from the first impulse to the second is equal to one half the orbital period of the transfer ellipse

$$\tau = \pi\sqrt{\frac{a^3}{\mu}} \tag{5}$$

where a is the semimajor axis of the transfer orbit and is equal to $\left(r_i + r_f\right)/2$. The orbital eccentricity of the transfer ellipse is

$$e = \frac{\max\left(r_i, r_f\right) - \min\left(r_i, r_f\right)}{r_f - r_i} \tag{6}$$

The following diagram illustrates the geometry of the coplanar Hohmann transfer.



Non-coplanar Analysis

The non-coplanar Hohmann transfer involves orbital transfer between two circular orbits which have different orbital inclinations. For this problem the propulsive energy is minimized if we optimally partition the total orbital inclination change between the first and second impulses.

The scalar magnitude of the first impulse is

$$\Delta V_1 = V_{lc}\sqrt{1 + R_1^2 - 2R_1\cos\theta_1} \tag{7}$$

where θ_1 is the plane change associated with the first impulse. The magnitude of the second impulse is

$$\Delta V_2 = V_{lc}\sqrt{R_2^2 + R_2^2 R_3^2 - 2R_2^2 R_3\cos\theta_2} \tag{8}$$

where θ_2 is the plane change associated with the second impulse. These two equations are different forms of the law of cosines.

The total ΔV required for the maneuver is the sum of the two impulses as follows

$$\Delta V = \Delta V_1 + \Delta V_2 \tag{9}$$

The relationship between the plane change angles is

$$\theta_t = \theta_1 + \theta_2 \tag{10}$$

where θ_t is the total plane change angle between the initial and final orbits.

Optimizing the non-coplanar Hohmann transfer involves allocating the total plane change angle between the two maneuvers such that the total ΔV required for the mission is minimized. We can determine this answer by solving for the root of a derivative.

The partial derivative of the total ΔV with respect to the first plane change angle is given by the following expression:

$$\frac{\partial\Delta V}{\partial\theta_1} = \frac{R_1\sin\theta_1}{\sqrt{1 + R_1^2 - 2R_1\cos\theta_1}} + \frac{R_2^2 R_3\left(\sin\theta_t\cos\theta_1 - \cos\theta_t\sin\theta_1\right)}{\sqrt{R_2^2 + R_2^2 R_3^2 - 2R_2^2 R_3\cos\left(\theta_t - \theta_1\right)}} \tag{11}$$

If we determine the value of θ_1 which makes this derivative zero, we have found the optimum plane change required at the first impulse. Once θ_1 is calculated we can determine θ_2 from the total plane change angle relationship. Finally, we can calculate the scalar magnitudes of the two velocity impulses from the equations (7) and (8).

Numerical Solution

The algorithm described here has been implemented in an interactive ***Numerit*** program called `hohmann`. This program prompts the user for the initial and final altitudes in kilometers and the initial and final orbital inclinations in degrees. The software then calls the Brent root-finding algorithm to solve the partial derivative equation described above.

The source code which accomplishes this is as follows:

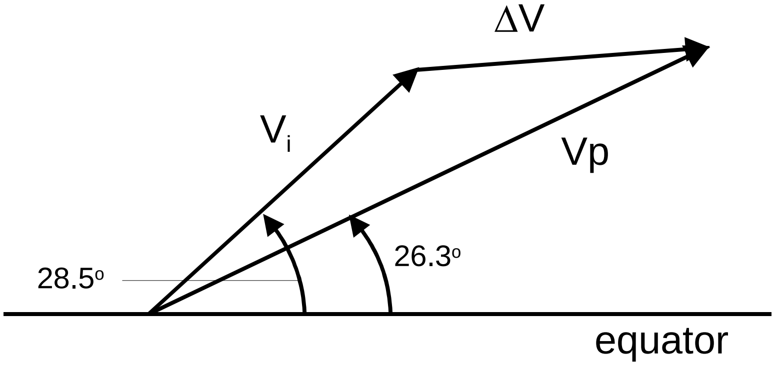
```
objfunc -> hohmfunc

brent(0, dinc, rtol, xroot, froot)
```

Since we know that the optimum first plane change angle is somewhere between 0 and the total plane change angle `dinc`, we pass these as the bounds of the root.

Let's illustrate this algorithm by solving a typical non-coplanar Hohmann orbit transfer from a low altitude Earth orbit (LEO) at an altitude of 185. 2 kilometers and an orbital inclination of 28.5 degrees to a geosynchronous Earth orbit (GEO) at an altitude of 35786.36 kilometers and 0 degrees inclination.

The following is a ΔV diagram for the <u>first</u> maneuver of this orbit transfer example. In this view we are looking along the line of nodes which is the mutual intersection of the park and transfer orbit planes with the equatorial plane.



In this diagram V_i is the speed on the initial park orbit, V_p is the perigee speed of the elliptic transfer orbit, and ΔV is the ΔV required for the first maneuver. The inclinations of the park and transfer orbit are also labeled. From this geometry and the law of cosines, the required ΔV is given by

$$\Delta V = \sqrt{V_i^2 + V_p^2 - 2V_i V_p \cos\Delta i} \tag{12}$$

where Δi is the inclination difference or plane change between the park and transfer orbits.

The following is a typical draft output created for this example.

```
Hohmann Orbit Transfer Analysis

initial orbit altitude                300 kilometers
initial orbit inclination            28.5 degrees
initial orbit velocity           7725.759 meters/second

final orbit altitude              35786.2 kilometers
final orbit inclination                 0 degrees
final orbit velocity             3074.654 meters/second

first inclination change         2.200202 degrees
```

```
second inclination change          26.2998 degrees
total inclination change              28.5 degrees

first delta-v                     2449.454 meters/second
second delta-v                    1781.853 meters/second
total delta-v                     4231.307 meters/second

transfer orbit eccentricity      0.7265438
transfer orbit perigee velocity   10151.49 meters/second
transfer orbit apogee velocity     1607.83 meters/second
```