Earth Orbit Lambert Problem

This ***Numerit*** program (`lambert1`) can be used to solve Lambert's problem for spacecraft in Earth orbit. Lambert's problem is concerned with the determination of an orbit which passes between two orbital positions within a specified time-of-flight. This classic astrodynamic problem is also known as the orbital *two-point boundary value problem* (TPBVP). The algorithm used in this software is valid for posigrade, retrograde and multi-revolution transfer orbits.

Lambert's Theorem

The time to traverse a trajectory depends only upon the length of the semimajor axis a of the transfer trajectory, the sum $r_i + r_f$ of the distances of the initial and final positions relative to the central body, and the length c of the chord joining these two positions.

Functionally, this geometric relationship can be stated as follows:

$$tof = tof\left(r_i + r_f, c, a\right) \tag{1}$$

From the following form of Kepler's equation

$$t - t_0 = \sqrt{\frac{a^3}{m}\left(E - e\sin E\right)} \tag{2}$$

we can write

$$t = \sqrt{\frac{a^3}{m}\left[E - E_0 - e\left(\sin E - \sin E_0\right)\right]} \tag{3}$$

where E is the eccentric anomaly associated with radius r, E_0 is the eccentric anomaly at r_0, and $t = 0$ when $r = r_0$.

At this point we need to introduce the following trigonometric *sum and difference identities*:

$$\sin a - \sin b = 2\sin\frac{a-b}{2}\cos\frac{a+b}{2}$$

$$\cos a - \cos b = -2\sin\frac{a-b}{2}\sin\frac{a+b}{2} \tag{4}$$

$$\cos a + \cos b = 2\cos\frac{a-b}{2}\cos\frac{a+b}{2}$$

If we let $E = a$ and $E_0 = b$ and substitute the first trig identity into the second equation above, we have the following equation:

$$t = \sqrt{\frac{a^3}{m}}\left\{E - E_0 - 2\sin\frac{E - E_0}{2}\left(e\cos\frac{E + E_0}{2}\right)\right\} \tag{5}$$

With the two substitutions given by

$$e\cos\frac{E + E_0}{2} = \cos\frac{a + b}{2}$$

$$\sin\frac{E - E_0}{2} = \sin\frac{a - b}{2}$$

the time equation becomes

$$t = \sqrt{\frac{a^3}{m}}\left\{(a - b) - 2\sin\frac{a - b}{2}\cos\frac{a + b}{2}\right\} \tag{6}$$

From the elliptic relationships given by

$$r = a(1 - e\cos E)$$
$$x = a(\cos E - e)$$
$$y = a\sin E\sqrt{1 - e^2}$$

and some more manipulation, we have the following equations:

$$\cos a = \left(1 - \frac{r + r_0}{2a}\right) - \frac{c}{2a} = 1 - \frac{r + r_0 + c}{2a} = 1 - \frac{s}{a}$$
$$\sin b = \left(1 - \frac{r + r_0}{2a}\right) + \frac{c}{2a} = 1 - \frac{r + r_0 - c}{2a} = 1 - \frac{s - c}{a} \tag{7}$$

This part of the derivation makes use of the following three relationships:

$$\cos\frac{a - b}{2}\cos\frac{a + b}{2} = 1 - \frac{r + r_0}{2}$$

$$\sin\frac{a - b}{2}\sin\frac{a + b}{2} = \sin\frac{E - E_0}{2}\sqrt{1 - \left(e\cos\frac{E + E_0}{2}\right)^2}$$

$$\left(\sin\frac{a - b}{2}\sin\frac{a + b}{2}\right)^2 = \left(\frac{x - x_0}{2a}\right)^2 + \left(\frac{y - y_0}{2a}\right)^2 = \left(\frac{c}{2a}\right)^2$$

With the use of the half angle formulas given by

$$\sin\frac{a}{2} = \sqrt{\frac{s}{2a}}$$

$$\sin\frac{b}{2} = \sqrt{\frac{s-c}{2a}}$$

and several additional substitutions, we have the *time-of-flight* form of Lambert's theorem

$$t = \sqrt{\frac{a^3}{m}}\left[(a-b) - (\sin a - \sin b)\right] \tag{8}$$

A discussion about the angles a and b can be found in "Geometrical Interpretation of the Angles a and b in Lambert's Problem" by J. E. Prussing, *AIAA Journal of Guidance and Control*, Volume 2, Number 5, Sept.-Oct. 1979, pages 442-443.

The algorithm used in this ***Numerit*** program is based on the method described in "A Practical Note on the Use of Lambert's Equation" by Geza Gedeon, *AIAA Journal*, Volume 3, Number 1, 1965, pages 149-150. This iterative solution is valid for elliptic, parabolic and hyperbolic transfer orbits which may be either posigrade or retrograde, and involve one or more revolutions about the central body. Additional information can also be found in G. S. Gedeon, "Lambertian Mechanics", *Proceedings of the 12th International Astronautical Congress*, Vol. I, 172-190.

The elliptic form of the general Lambert Theorem is

$$t = \sqrt{\frac{a^3}{m}}\left[(1-k)mp + k(a - \sin a) \mp (b - \sin b)\right] \tag{9}$$

where k may be either $+1$ (posigrade) or -1 (retrograde), and m is the number of revolutions about the central body.

The Gedeon algorithm introduces the following variable

$$z = \frac{s}{2a}$$

and solves the problem with a Newton-Raphson iterative procedure. In this equation, a is the semimajor axis of the transfer orbit and

$$s = \frac{r_1 + r_2 + c}{2}$$

This algorithm also makes use of the following geometric constant

$$w = \pm\sqrt{1 - \frac{c}{s}}$$

The function to be solved iteratively is given by

$$N(z) = \frac{1}{z|z|^{1/2} 2^{1/2}} \left\{ \frac{1-k}{2} mp + k\left[|z|^{1/2} - |z|^{1/2} (1-z)^{1/2} \right] - \left[w|z|^{1/2} (1 - w^2 z)^{1/2} \right] \right\} \quad (10)$$

The Newton-Raphson algorithm also requires the derivative of this equation which is given by the following expression:

$$N'(z) = \frac{dN}{dz} = \frac{1}{|z| 2^{1/2}} \left\{ \frac{k}{(1-z)^{1/2}} - \frac{w^3}{(1-w^2 z)^{1/2}} - \frac{3N(z)}{2^{1/2}} \right\} \quad (11)$$

The iteration for z is as follows:

$$z_{n+1} = z_n - \frac{N(z_n)}{N'(z_n)} \quad (12)$$

The syntax and function arguments for this function are as follows:

```
function lambfunc(ri, rf, tof, direct, revmax, statev, nsol)

` solve Lambert's orbital two point boundary value problem

` input

`  ri     = initial ECI position vector (kilometers)
`  rf     = final ECI position vector (kilometers)
`  tof    = time of flight (seconds)
`  direct = transfer direction (1 = posigrade, -1 = retrograde)
`  revmax = maximum number of complete orbits

` output

`  nsol   = number of solutions
`  statev = matrix of solutions
```

The Lambert inertial state vector and orbital elements at the *initial* time are returned in the two-dimensional `statev` array which is organized as follows:

```
statev(1, sn) = position vector x component
statev(2, sn) = position vector y component
statev(3, sn) = position vector z component
statev(4, sn) = velocity vector x component
statev(5, sn) = velocity vector y component
```

```
statev(6, sn) = velocity vector z component
statev(7, sn) = semimajor axis
statev(8, sn) = orbital eccentricity
statev(9, sn) = orbital inclination
statev(10, sn) = argument of perigee
statev(11, sn) = right ascension of the ascending node
statev(12, sn) = true anomaly
```

In this array `sn` is the solution number.

The software will ask for the classical orbital elements of the initial and final orbits. It will also ask you to input the transfer time in hours, the transfer direction and the number of revolutions about the Earth.

The following is a typical draft output created with this software.

```
program lambert1

< Earth orbit lambert problem >

orbital elements of the initial orbit

     sma (km)         eccentricity    inclination (deg)    argper (deg)
          8000                0           28.5                    0

    raan (deg)     true anomaly (deg)   arglat (deg)       period (min)
           100               45               45       118.6846843

orbital elements of the final orbit

     sma (km)         eccentricity    inclination (deg)    argper (deg)
         10000            0.015           40                  200

    raan (deg)     true anomaly (deg)   arglat (deg)       period (min)
            55               10              210       165.86688873

solution            1

orbital elements of the transfer orbit

     sma (km)         eccentricity    inclination (deg)    argper (deg)
   9200.3720641      0.13131269296    22.216585744         109.42160214

    raan (deg)     true anomaly (deg)   arglat (deg)       period (min)
   22.658311849      7.4094175579      116.8310197         146.37529533

delta-v requirements

x-component of delta-v       -1601.103713 meters/second
y-component of delta-v       269.9812694 meters/second
z-component of delta-v       -3624.374361 meters/second

total delta-v                3971.462261 meters/second

transfer time                       0.75 hours
```