

Optimal, Continuous Low-thrust Coplanar Orbit Transfer

This *Numerit* program (ocot) solves a classic problem in optimal control theory. The complete problem is described on pages 66-69 of the classic text *Applied Optimal Control*, by Arthur E. Bryson, Jr. and Yu-Chi Ho, Hemisphere Publishing Company, Washington, D.C., 1975.

This problem is summarized in the text as follows:

"Given a constant-thrust rocket engine, T = thrust, operating for a given length of time, t_f , we wish to find the thrust-direction history, $\mathbf{f}(t)$, to transfer a rocket vehicle from a given initial circular orbit to the largest possible circular orbit."

In the language of optimal control theory, this problem is a "continuous system with functions of the state variables prescribed at a fixed terminal time".

The first-order, two-dimensional equations of motion for this problem are given by

$$\begin{aligned}\dot{r} &= u \\ \dot{u} &= \frac{v^2}{r} - \frac{\mathbf{m}}{r^2} + \frac{T}{m} \sin \mathbf{f} \\ \dot{v} &= -\frac{uv}{r} + \frac{T}{m} \cos \mathbf{f}\end{aligned}\tag{1}$$

where

- r = radial position
- u = radial velocity
- v = transverse velocity
- T = propulsive thrust
- m = spacecraft mass
- \mathbf{f} = thrust angle
- \mathbf{m} = gravitational constant of the central body

The spacecraft mass at any time t is determined from

$$m(t) = m_0(1 - \dot{m}t)\tag{2}$$

where \dot{m} is the (assumed constant) propellant flow rate and m_0 is the mass of the spacecraft at the initial time.

The numerical example given in the text is a coplanar orbit transfer from Earth to Mars. The orbit of each planet is assumed circular and the transfer time is about 193 days.

The initial conditions of the spacecraft are summarized as follows:

$$\begin{aligned}
 r(0) &= r_0 \\
 u(0) &= 0 \\
 v(0) &= \sqrt{\frac{m}{r_0}}
 \end{aligned} \tag{3}$$

The *boundary conditions* at the final time t_f are given by

$$\begin{aligned}
 u(t_f) &= 0 \\
 v(t_f) - \sqrt{\frac{m}{r(t_f)}} &= 0
 \end{aligned} \tag{4}$$

The scalar *Hamiltonian* for this dynamical system is

$$H = \mathbf{l}_r u + \mathbf{l}_u \left(\frac{v^2}{r} - \frac{m}{r^2} + \frac{T}{m} \sin \mathbf{f} \right) + \mathbf{l}_v \left(-\frac{uv}{r} + \frac{T}{m} \cos \mathbf{f} \right) \tag{5}$$

Formulating the Euler-Lagrange equations and enforcing the *necessary conditions* for the optimal results in the following *adjoint* differential equations:

$$\begin{aligned}
 \mathbf{l}_r &= -\mathbf{l}_u \left(-\frac{v^2}{r^2} + \frac{2m}{r^3} \right) - \mathbf{l}_v \left(\frac{uv}{r^2} \right) \\
 \mathbf{l}_u &= -\mathbf{l}_r + \mathbf{l}_v \frac{v}{r} \\
 \mathbf{l}_v &= -\mathbf{l}_u \frac{2v}{r} + \mathbf{l}_v \frac{u}{r}
 \end{aligned} \tag{6}$$

In the terminology of the calculus of variations, the adjoints are also called the *costates* or *Lagrange multipliers*.

The optimal thrusting angle is defined by

$$\tan \mathbf{f} = \frac{\mathbf{l}_u}{\mathbf{l}_v} \tag{7}$$

This angle is measured relative to the "local horizontal" or tangential direction at the spacecraft's location. It is positive above the local horizontal and negative below.

At the final time t_f the radial costate satisfies the following *transversality* condition:

$$\mathbf{I}_r(t_f) = 1 + \frac{\mathbf{I}_v(t_f)\sqrt{\mathbf{m}}}{2\{r(t_f)\}^{3/2}} \quad (8)$$

This computer program attempts to solve the boundary conditions given by Equations (4) and (8) while solving the system of differential equations given by Equations (1) and (6). At each time step of the orbit transfer the optimal thrust or *steering* angle is determined from Equation (7). The solution of an optimal control problem subject to initial and final boundary conditions is called a *two-point boundary-value problem*. The boundary conditions are solved as a system of three nonlinear equations.

The initial mass and propulsive characteristics for the Earth to Mars orbit transfer example are as follows:

initial thrust $T = 3.781$ newtons
 initial spacecraft mass $m_0 = 4535.9$ kilograms
 propellant flow rate $\dot{m} = 5.85$ kilograms/day

The *non-dimensional* acceleration due to thrusting is given by

$$\frac{T/m_0}{\mathbf{m}/r_0^2} = 0.1405 \quad (9)$$

The non-dimensional *acceleration unit* (AU) is \mathbf{m}/r_0^2 .

The *non-dimensional* total flight time is given by

$$\frac{t_f}{\sqrt{r_0^3/\mathbf{m}}} = 3.32 \quad (10)$$

The non-dimensional *time unit* (TU) used in the software is $\sqrt{r_0^3/\mathbf{m}}$.

The non-dimensional value of the gravitational constant \mathbf{m} is 1. The dimensional value of the Sun's gravitational constant is $132712441933 \text{ km}^3/\text{sec}^2$. The non-dimensional value of the initial distance r_0 is 1. The dimensional value of the radius of the Earth's orbit r_0 is 149597870 kilometers or one Astronomical Unit.

From the dimensional propellant flow rate and Equation (2) we can determine the non-dimensional propellant flow rate which is equal to -0.07487. This orbit transfer requires about 1129 kilograms of propellant.

Orbital Mechanics with Numerit

The following is the draft output for this example.

```
program ocot
< optimal, continuous low-thrust coplanar orbit transfer >
adjoint variables (costates) initial guesses
lambda r  1
lambda u  1
lambda v  1
initial conditions (non-dimensional)
radial position      1
radial velocity      0
transverse velocity  1
adjoint variables (costates) solution
lambda r  1.87730104174674
lambda u  0.928938649809026
lambda v  2.02507901228801
final coordinates (non-dimensional)
radial position      1.52524627971771
radial velocity      8.05321859545516e-11
transverse velocity  0.809710950729154
transfer angle      142.623323987934  degrees
```

The following is a plot of the thrust angle during the orbit transfer.

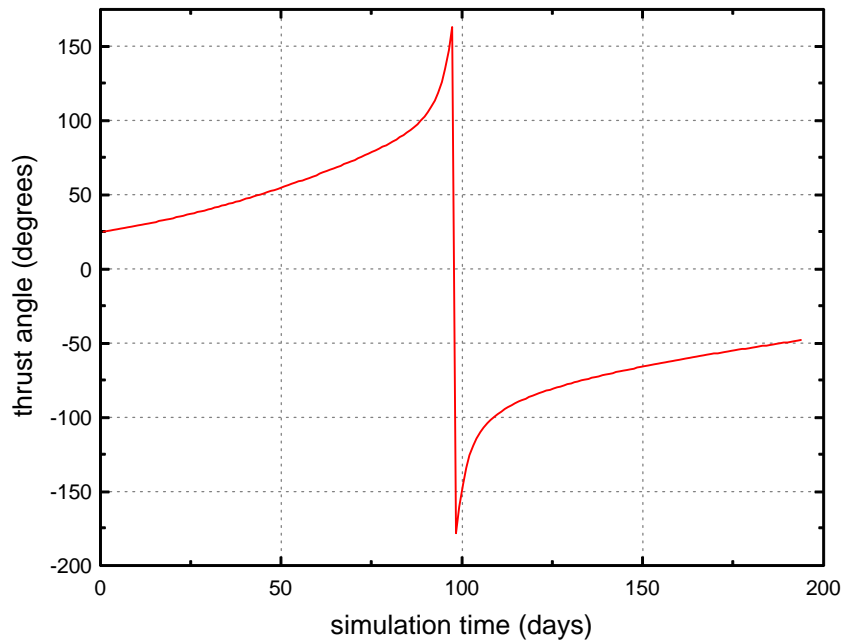


Figure 1. Thrust Angle versus Time

The following is a plot of the transfer trajectory.

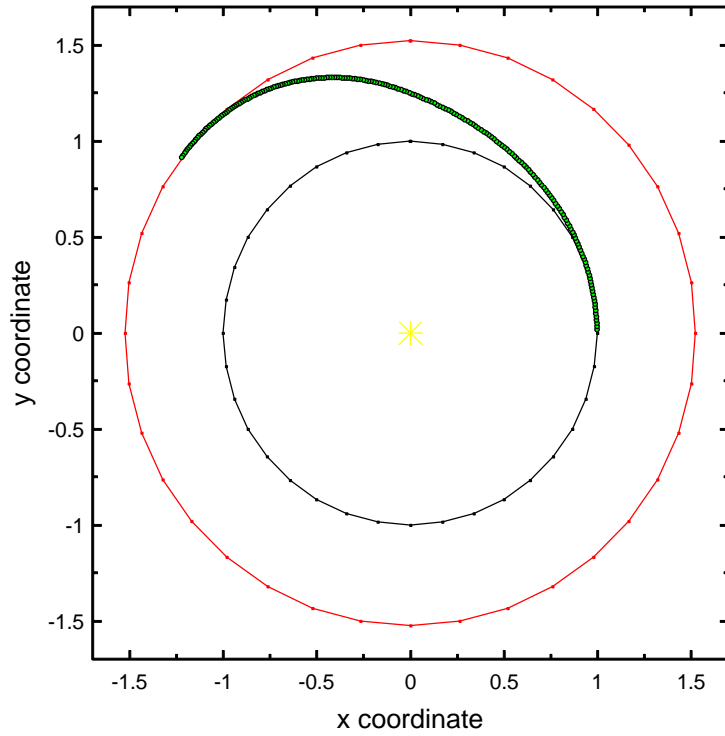


Figure 2. Transfer Trajectory

This next plot illustrates the behavior of the costates during the orbit transfer.

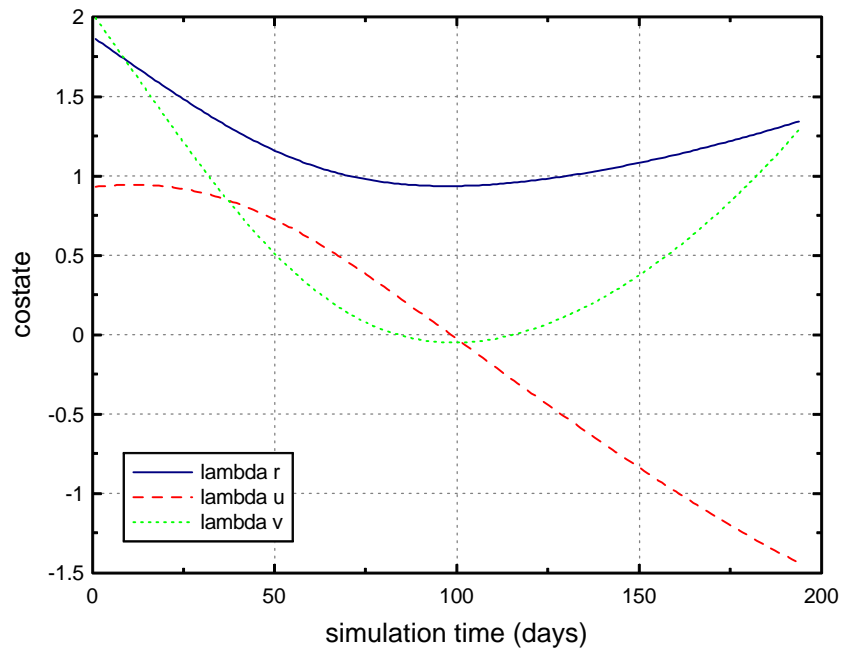


Figure 3. Costates versus Time