Primer Vector Analysis of Coplanar Hohmann Transfers

This ***Numerit*** program (`primer`) demonstrates how to use *primer vector theory* to analyze the performance of impulsive orbital transfers. The term primer vector was invented by Derek F. Lawden and represents the adjoint vector for velocity. A technical discussion about primer theory can be found in Professor Lawden's classic text, *Optimal Trajectories for Space Navigation*, Butterworths, London, 1963. Another excellent resource is "Primer Vector Theory and Applications", by Donald J. Jezewski, NASA TR R-454, November 1975, along with "Optimal, Multi-burn, Space Trajectories", also by Donald Jezewski..

As shown by Lawden, the following four necessary conditions must be satisfied in order for an impulsive orbital transfer to be *locally optimal*:

(1) the primer vector and its first derivative are everywhere continuous

(2) whenever a velocity impulse occurs, the primer vector is aligned with the impulse and has unit magnitude $\left(\boldsymbol{p} = \hat{\boldsymbol{p}} = \hat{\boldsymbol{u}}_T \text{ and } \|\boldsymbol{p}\| = 1\right)$

(3) the magnitude of the primer vector may not exceed unity on a coasting arc $\left(\|\boldsymbol{p}\| = p \le 1\right)$

(4) at all interior impulses (not at the initial or final time) $\boldsymbol{p} \bullet \dot{\boldsymbol{p}} = 0$; therefore, $d\|\boldsymbol{p}\|/dt = 0$ at the intermediate impulses

Furthermore, the scalar magnitudes of the primer vector derivative at the initial and final impulses provide information about how to improve the nominal transfer trajectory by changing the endpoint times and/or moving the impulse times. These four cases for non-zero slopes are summarized as follows;

(1) If $\dot{p}_0 > 0$ and $\dot{p}_f < 0$, then perform an initial coast before the first impulse and add a final coast after the second impulse

(2) If $\dot{p}_0 > 0$ and $\dot{p}_f > 0$, then perform an initial coast before the first impulse and move the second impulse to a later time

(3) If $\dot{p}_0 < 0$ and $\dot{p}_f < 0$, then perform the first impulse at an earlier time and add a final coast after the second impulse

(4)If $\dot{p}_0 < 0$ and $\dot{p}_f > 0$, then perform the first impulse at an earlier time and move the second impulse to a later time

The primer vector analysis of a two impulse orbital transfer involves the following steps.

First partition the state transition matrix as follows:

$$\Phi(t, t_0) = \begin{bmatrix} \dfrac{\partial \boldsymbol{r}}{\partial \boldsymbol{r}_0} & \dfrac{\partial \boldsymbol{r}}{\partial \mathbf{v}_0} \\ \dfrac{\partial \mathbf{v}}{\partial \boldsymbol{r}_0} & \dfrac{\partial \mathbf{v}}{\partial \mathbf{v}_0} \end{bmatrix} = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \tag{1}$$

where

$$\Phi_{11} = \left[\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{r}_0} \right] = \begin{bmatrix} \partial x / \partial x_0 & \partial x / \partial y_0 & \partial x / \partial z_0 \\ \partial y / \partial x_0 & \partial y / \partial y_0 & \partial y / \partial z_0 \\ \partial z / \partial x_0 & \partial z / \partial y_0 & \partial z / \partial z_0 \end{bmatrix}$$

and so forth.

The value of the primer vector at any point along a two body trajectory is given by

$$\boldsymbol{p}(t) = \Phi_{11}(t, t_0)\boldsymbol{p}_0 + \Phi_{11}(t, t_0)\dot{\boldsymbol{p}}_0 \tag{2}$$

and the value of the primer vector derivative is

$$\dot{\boldsymbol{p}}(t) = \Phi_{21}(t, t_0)\boldsymbol{p}_0 + \Phi_{22}(t, t_0)\dot{\boldsymbol{p}}_0 \tag{3}$$

which can also be expressed as

$$\begin{Bmatrix} \boldsymbol{p} \\ \dot{\boldsymbol{p}} \end{Bmatrix} = \Phi(t, t_0) \begin{Bmatrix} \boldsymbol{p}_0 \\ \dot{\boldsymbol{p}}_0 \end{Bmatrix} \tag{4}$$

The primer vector boundary conditions at the initial and final impulses are as follows:

$$\begin{aligned} \boldsymbol{p}(t_0) &= \boldsymbol{p}_0 = \frac{\Delta \mathbf{v}_0}{|\Delta \mathbf{v}_0|} \\ \boldsymbol{p}(t_f) &= \boldsymbol{p}_f = \frac{\Delta \mathbf{v}_f}{|\Delta \mathbf{v}_f|} \end{aligned} \tag{5}$$

These two conditions illustrate that at the locations of velocity impulses, the primer vector is a unit vector in the direction of the impulses.

Finally, the value of the primer vector derivative at the initial time is

$$\dot{\boldsymbol{p}}(t_0) = \dot{\boldsymbol{p}}_0 = \Phi_{12}^{-1}(t, t_0)\left\{ \boldsymbol{p}_f - \Phi_{11}(t, t_0)\boldsymbol{p}_0 \right\} \tag{6}$$

provided the Φ_{12} sub-matrix is non-singular.

The scalar magnitude of the derivative of the primer vector can be determined from

$$\frac{d\|\boldsymbol{p}\|}{dt} = \frac{d}{dt}(\boldsymbol{p} \bullet \boldsymbol{p})^{2} = \frac{\dot{\boldsymbol{p}} \bullet \boldsymbol{p}}{\|\boldsymbol{p}\|} \tag{7}$$

This ***Numerit*** program creates plots of the scalar magnitudes of the primer vector and its derivative for a two impulse, coplanar Hohmann transfer. It will request the altitudes of the initial and final circular orbits.

The following is a typical draft output created with this software.

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program primer

< primer vector analysis of coplanar Hohmann transfers >


initial orbit altitude                 185.2 kilometers

initial orbit inclination                  0 degrees

initial orbit velocity             7793.0337 meters/second


final orbit altitude                   35790 kilometers

final orbit inclination                    0 degrees

final orbit velocity               3074.5155 meters/second


first delta-v                      2458.9755 meters/second

second delta-v                     1478.8228 meters/second

total delta-v                      3937.7984 meters/second


transfer orbit eccentricity       0.73063255

transfer orbit perigee velocity    10252.009 meters/second

transfer orbit apogee velocity     1595.6926 meters/second


transfer time-of-flight            18925.628 seconds
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The following are plots of the scalar magnitude of the primer vector and primer derivative as a function of elapsed trajectory time for this coplanar orbital transfer. From these plots and the necessary conditions for optimality, we can see that this is a locally optimal two impulse transfer trajectory since $p_0 = p_f = 1$.

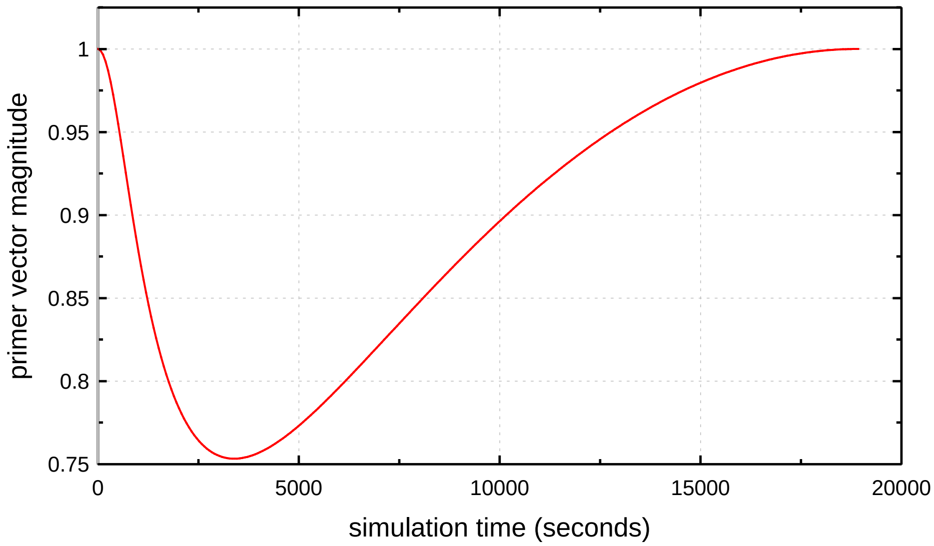

Figure 1. Primer Vector Magnitude

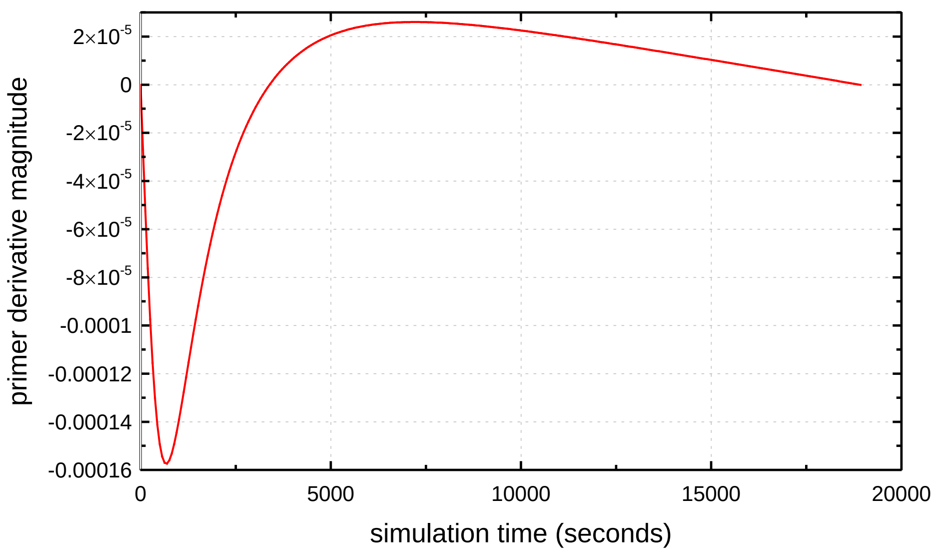

Figure 2. Primer Derivative Magnitude

From this plot of the primer derivative magnitude $\left(\dot{p}_0 = \dot{p}_f \approx 0\right)$ and the rules for moving impulses or adding coasts, no modifications of this trajectory will reduce the total delta-v.