Shadow Conditions of Satellites in Circular Earth Orbits

The approximate eclipse duration of a satellite in a circular orbit which penetrates an Earth shadow represented by a right circular cylinder can be calculated from

$$t_s = \left\{ \cos^{-1}\left(\frac{\sqrt{1-R^2}}{\cos\beta} \right) \right\} \frac{\tau}{\pi} \tag{1}$$

where

$R =$ radius ratio $= r_{eq} / r_{sat}$
$r_{eq} =$ equatorial radius of the Earth
$r_{sat} =$ geocentric radius of the satellite $= r_{eq} + h_{sat}$
$h_{sat} =$ altitude of the satellite
$\beta =$ Sun-orbit-plane angle
$\tau =$ orbital period of the satellite $= 2\pi\sqrt{r_{sat}^3 / \mu}$
$\mu =$ gravitational constant of the Earth

Equation (1) also assumes that the Sun does not move during the eclipse. Furthermore, the time units of eclipse duration are the same as the orbital period τ provided the inverse cosine in this equation returns an angle in radians.

The part of Equation (1) represented by

$$\cos^{-1}\left(\frac{\sqrt{1-R^2}}{\cos\beta} \right)$$

is one half the true anomaly angle traversed by the satellite during the eclipse. A closer examination of this equation also reveals that satellites at altitudes which satisfy the radius ratio inequality given by

$$R > \sin\left|\beta_{\max}\right|$$

will have periods during which they are not eclipsed by the Earth.

The Sun-orbit-plane or *beta* angle is the angle between the geocentric position vector to the Sun and the satellite's orbit plane. It is calculated from

$$\beta = \sin^{-1}\left(\hat{\boldsymbol{r}}_{sun} \bullet \hat{\boldsymbol{h}}_{sat} \right) \tag{2}$$

where $\hat{\boldsymbol{r}}_{sun}$ is the geocentric unit position vector of the Sun and $\hat{\boldsymbol{h}}_{sat}$ is the unit angular momentum vector of the satellite's orbit. The unit angular momentum vector is defined by the cross product $\hat{\boldsymbol{r}}_{sat} \times \hat{\boldsymbol{v}}_{sat}$ where $\hat{\boldsymbol{r}}_{sat}$ and $\hat{\boldsymbol{v}}_{sat}$ are the unit position and velocity vectors of the satellite, respectively. A positive beta angle indicates that the Sun in on the positive angular momentum side of the orbit plane. The inertial orientation of this vector changes over time as the oblateness of the Earth causes the orbit plane to move.

The unit angular momentum vector can also be determined from a satellite's classical orbital elements with

$$\hat{\boldsymbol{h}}_{sat} = \begin{Bmatrix} \sin\Omega\sin i \\ -\cos\Omega\sin i \\ \cos i \end{Bmatrix} \tag{3}$$

where i is the satellite's orbital inclination and Ω is the right ascension of the ascending node (RAAN).

The unit position vector of the Sun can be determined with the expression

$$\hat{\boldsymbol{r}}_{sun} = \begin{Bmatrix} \cos\delta_{sun}\cos\alpha_{sun} \\ \cos\delta_{sun}\sin\alpha_{sun} \\ \sin\delta_{sun} \end{Bmatrix} \tag{4}$$

where α_{sun} and δ_{sun} are the geocentric, equatorial right ascension and declination of the Sun, respectively.

The beta angle in terms of the satellite's RAAN and orbital inclination is given by

$$\beta = \sin^{-1}\left(\cos\delta_{sun}\sin i\sin\left(\Omega - \alpha_{sun}\right) + \sin\delta_{sun}\cos i\right) \tag{5}$$

Over a period of one year the beta angle varies between $-\left(i + \delta_{sun}\right) \le \beta \le +\left(i + \delta_{sun}\right)$ as the solar declination varies between about plus and minus 23.5°.

An examination of Equation (1) reveals that the maximum shadow time occurs when $\cos\beta = 1$ which corresponds to $\beta = 0$. This is the instant when the Sun unit position vector $\hat{\boldsymbol{r}}_{sun}$ lies in the satellite's orbit plane. The minimum shadow time will occur when $\cos\beta$ reaches its minimum value. This happens when the beta angle reaches its largest plus and minus values.

During a simulation the value of RAAN at the initial time t_0 is updated for the secular effect of Earth oblateness according to

$$\Omega(t) = \Omega_0 + \dot{\Omega}\left(t - t_0\right) \tag{6}$$

where

$$\dot{\Omega} = \frac{d\Omega}{dt} = -\frac{3}{2} J_2 \tilde{n} \left(\frac{r_{eq}}{p}\right)^2 \cos i$$

These equations have been implemented in a ***Numerit*** program called `shadow1`. This software will prompt the user for an initial calendar date, a satellite's orbital elements (altitude, inclination, RAAN), and a simulation duration and orbit propagation step size. The software will calculate the minimum and maximum values of the beta angle, and the minimum, maximum and average shadow duration. This computer program also creates data arrays which are graphically displayed in the companion Report for this software.

The following is a typical data summary created with this program.

```
Shadow Conditions of Earth Satellites

circular orbit altitude           350 kilometers
orbital inclination              28.5 degrees
RAAN                              100 degrees
Keplerian period             91.53817 minutes

minimum beta angle          -45.47706 degrees
maximum beta angle           48.93324 degrees

minimum shadow duration      34.15341 minutes
maximum shadow duration      38.25584 minutes
average shadow duration      37.48425 minutes
```

The shadow duration and beta angle relationships are illustrated in the following figures for a satellite at an altitude of 350 kilometers, an inclination of 28.5° and an initial RAAN of 100°.

Figure 1 is a plot of the shadow duration as a function of simulation time and Figure 2 is the beta angle. For this example the simulation duration is 180 days, the step size is 60 minutes and the initial calendar date is January 1, 1999.

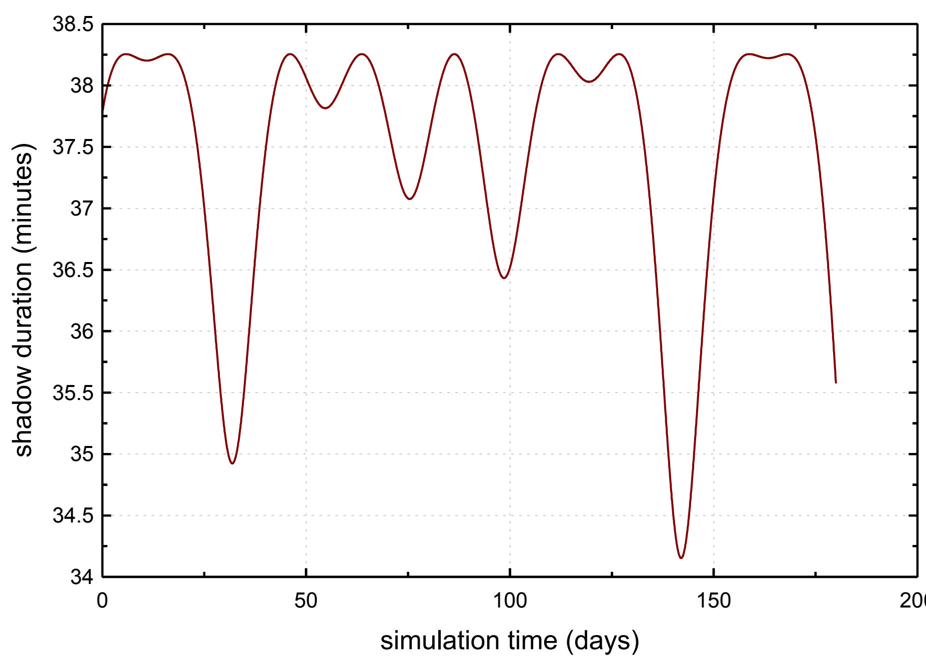

Figure 1. Shadow Duration

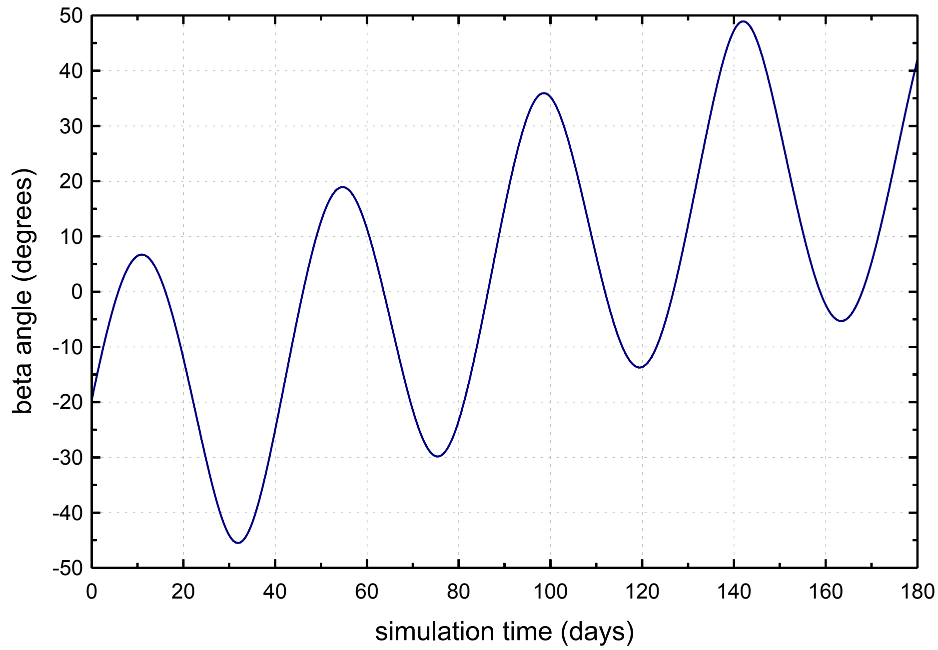

Figure 2. Beta Angle

We can also examine the data used to create these plots using the ***Numerit*** Viewer-Table capability. Here are the first ten rows of information for this example.

time (days)	duration (min)	beta (deg)
0	37.78	-19.66
0.042	37.79	-19.5
0.083	37.8	-19.33
0.12	37.8	-19.17
0.17	37.81	-19
0.21	37.82	-18.84
0.25	37.83	-18.67
0.29	37.84	-18.51
0.33	37.84	-18.34
0.38	37.85	-18.18