Parametric Analysis of Minimum TLI Delta-V, Two-Body Lunar Transfer Trajectories

This ***Numerit*** program (`tli_sweep`) can be used to perform a parametric analysis of two-body Earth-to-Moon transfer trajectories. The software assumes that translunar injection (TLI) occurs impulsively from a circular Earth park orbit using a single maneuver. The software solves for the impulsive TLI delta-v using a two-body Lambert solution for the transfer trajectory from the Earth park orbit to the center of the moon. It is useful for preliminary analysis of translunar trajectories and can also create initial guesses for more sophisticated computer programs.

The ephemeris of the Moon is computed using the SLP96 precision ephemeris. The two-body Lambert solution is based on Gedeon's algorithm.

Input data file

The `tli_sweep` computer program is "data-driven" by a simple text file created by the user. This section describes a typical input data file. In the following discussion the actual input file contents are in courier font and all explanations are in times italic font.

Each data item within an input file is preceded by one or more lines of annotation text. Do not delete any of these annotation lines or increase or decrease the number of lines reserved for each comment. However, you may change them to reflect your own explanation. The annotation line also includes the correct units and when appropriate, the valid range of the input.

The first four lines of any input file are reserved for user comments. These lines are ignored by the software. However the input file must begin with four and only four initial text lines.

```
************************************
* data file for tli_sweep program
* tli_sweep1.in    December 20, 2007
************************************
```

The first inputs define the calendar date of the TLI maneuver. Be sure to include all four digits of the calendar year. The day value can include a fractional part as well.

```
initial calendar date (month, day, year)
1, 1, 2008
```

The next input specifies the type of TLI maneuver. Please see the Technical Discussion later in this document for an explanation of this maneuver.

```
type of TLI maneuver (1 = ascending, 2 = descending)
2
```

The next two inputs define the value of the altitude and orbital inclination of the circular park orbit.

```
park orbit altitude (kilometers)
185.2

park orbit inclination (degrees)
28.5
```

The duration of the lunar transfer trajectory is set by this next input.

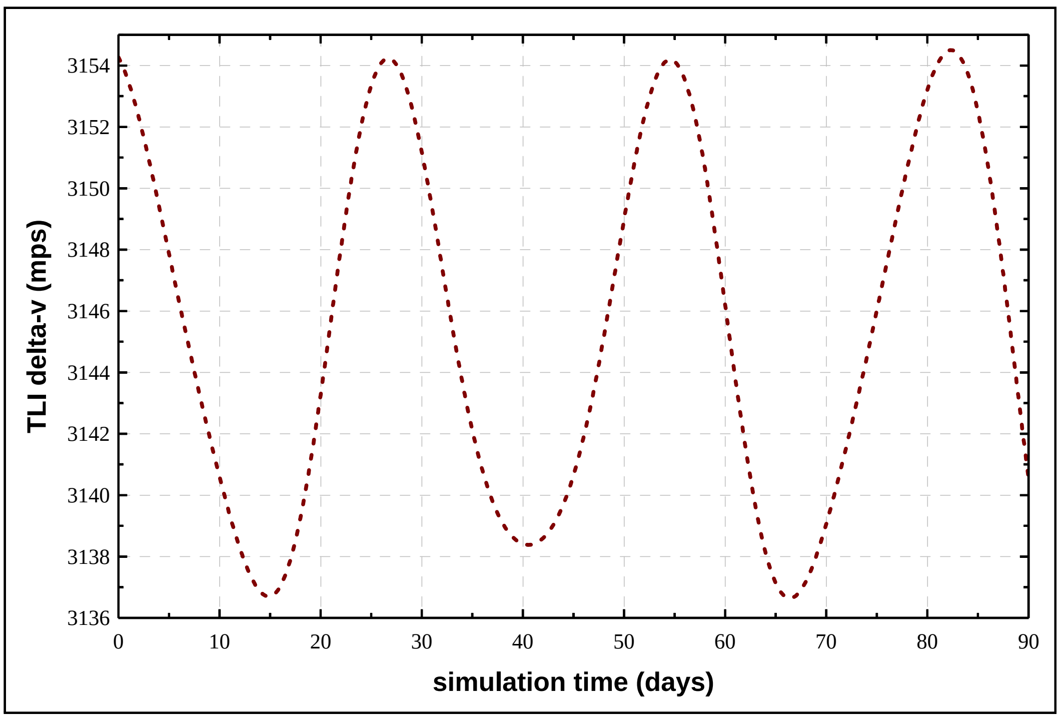
```
transfer time (hours)
84.0
```

The total simulation duration and time step size of the parametric sweep are specified using these next two inputs.

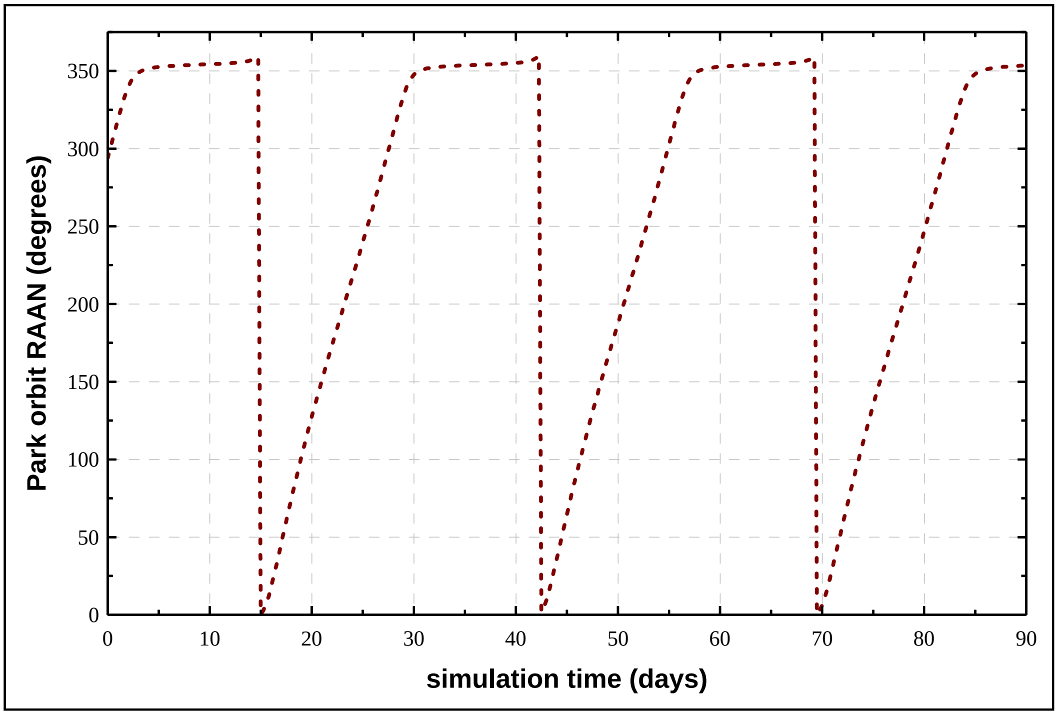
```
simulation duration (days)
90

simulation step size (days)
0.25
```

Program example

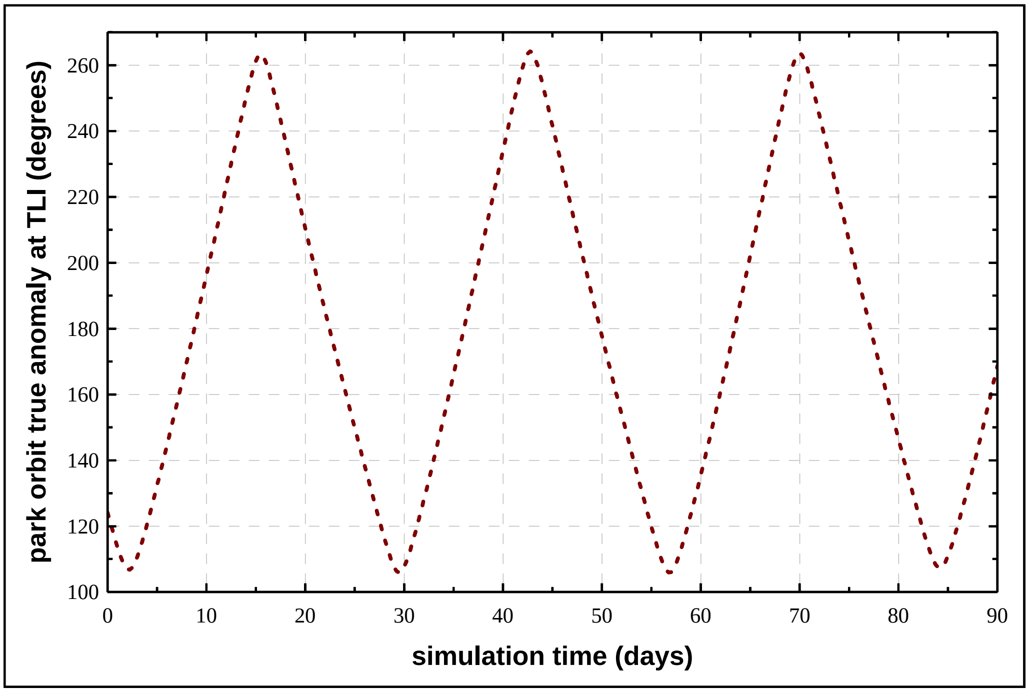
This section contains plots of the behavior of the TLI characteristics for a descending transfer for a period of three months at a step size of 0.25 days. The initial calendar date and time is 0 hours TDB on January 1, 2008 and the transfer time from TLI until lunar encounter is 84 hours. The first plot shows the variation of the magnitude of the TLI impulsive delta-v as a function of the elapsed time from the initial epoch.



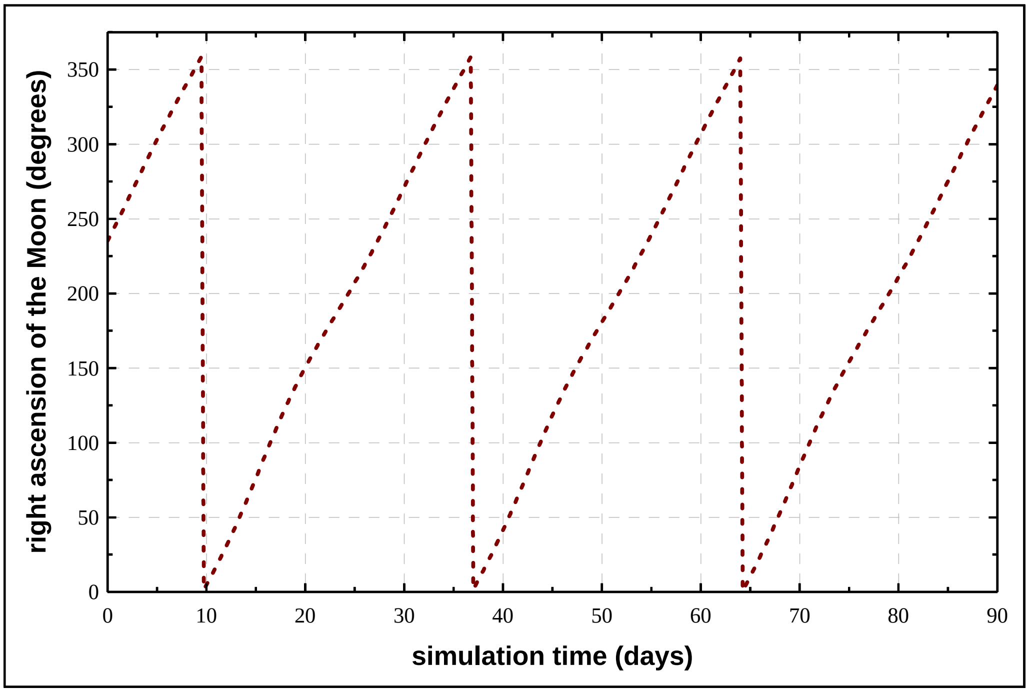
This next plot illustrates the variation of the RAAN of the circular park orbit as a function of the TLI calendar date.

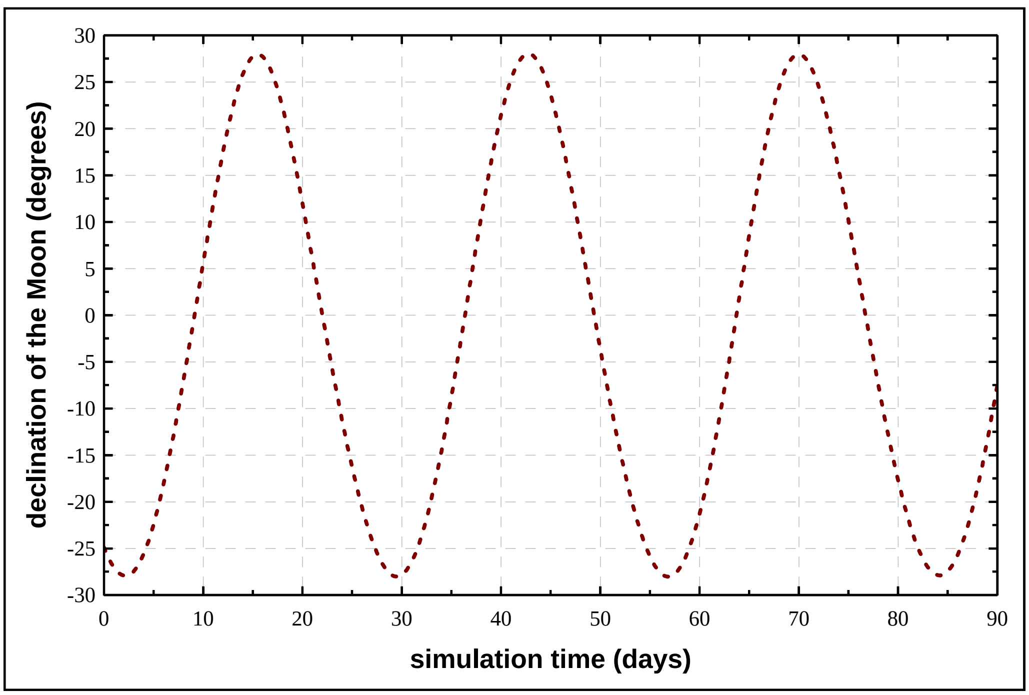


This plot illustrates the behavior of the true anomaly of the impulsive TLI maneuver on the circular park orbit.



The next two plots illustrate the geocentric right ascension and declination of the Moon during this simulation period.





Please note that all angular coordinates are measured with respect to the Earth mean equator and equinox of J2000 (EME2000) coordinate system.

Technical Discussion

This section describes several of the major numerical methods implemented in the `tli_sweep` computer program.

Park orbit true anomaly

The park orbit true anomaly at the time of the impulsive TLI maneuver is obtained iteratively. This iteration involves setting the park orbit true anomaly equal to the argument of latitude of the TLI maneuver on the Earth-to-Moon elliptical transfer orbit. This process is repeated until the change in true anomaly between successive iterations is small. The argument of latitude on the transfer orbit is determined by resolving the two-body Lambert problem which is a function of the current park orbit position vector and the (fixed) position vector of the Moon at encounter.

Park orbit RAAN

For a given TLI calendar date, there are two possible locations on the initial park orbit at which to perform the propulsive maneuver. One opportunity occurs during the ascending part of the park orbit and the other during the descending motion. The park orbit RAAN at these two locations can be determined from spherical trigonometry relationships involving the park orbit inclination and the geocentric right ascension and declination of the moon at encounter. The RAAN option used is selected by the user.

The equations implemented in this ***Numerit*** program are as follows:

ascending maneuver

$$\Omega_p = -180° + \alpha_m + \sin^{-1}\left(\frac{\tan\delta_m}{\tan i_p}\right) \tag{1}$$

descending maneuver

$$\Omega_p = \alpha_m + \sin^{-1}\left(\frac{\tan\delta_m}{\tan i_p}\right) \tag{2}$$

where

$$\begin{aligned} \alpha_m &= \text{ right ascension of the Moon at encounter} \\ \delta_m &= \text{ declination of the Moon at encounter} \\ i_p &= \text{ park orbit inclination} \end{aligned} \tag{3}$$

Solving the two body Lambert problem

Lambert's problem is concerned with the determination of an orbit that passes between two positions within a specified time-of-flight. This classic astrodynamic problem is also known as the orbital two-point boundary value problem (TPBVP).

The time to traverse a trajectory depends only upon the length of the semimajor axis a of the transfer trajectory, the sum $r_i + r_f$ of the distances of the initial and final positions relative to the central body, and the length c of the chord joining these two positions.

Functionally, this geometric relationship can be stated as follows:

$$tof = tof\left(r_i + r_f,\, c,\, a\right) \tag{4}$$

From the following form of Kepler's equation

$$t - t_0 = \sqrt{\frac{a^3}{\mu}}\left(E - e\sin E\right) \tag{5}$$

we can write

$$t = \sqrt{\frac{a^3}{\mu}}\left[E - E_0 - e\left(\sin E - \sin E_0\right)\right] \tag{6}$$

where E is the eccentric anomaly associated with radius r, E_0 is the eccentric anomaly at r_0, and $t = 0$ when $r = r_0$.

At this point we need to introduce the following trigonometric *sum and difference identities*:

$$\begin{aligned} \sin\alpha - \sin\beta &= 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2} \\ \cos\alpha - \cos\beta &= -2\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2} \\ \cos\alpha + \cos\beta &= 2\cos\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2} \end{aligned} \tag{7}$$

If we let $E = \alpha$ and $E_0 = \beta$ and substitute the first trig identity into the second equation above, we have the following equation:

$$t = \sqrt{\frac{a^3}{\mu}}\left\{E - E_0 - 2\sin\frac{E - E_0}{2}\left(e\cos\frac{E + E_0}{2}\right)\right\} \tag{8}$$

With the two substitutions given by

$$\begin{aligned} e\cos\frac{E + E_0}{2} &= \cos\frac{\alpha+\beta}{2} \\ \sin\frac{E - E_0}{2} &= \sin\frac{\alpha-\beta}{2} \end{aligned}$$

the time equation becomes

$$t = \sqrt{\frac{a^3}{\mu}}\left\{(\alpha - \beta) - 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}\right\} \tag{9}$$

From the elliptic relationships given by

$$r = a(1 - e\cos E)$$
$$x = a(\cos E - e)$$
$$y = a\sin E\sqrt{1-e^2}$$

and some more manipulation, we have the following equations:

$$\begin{aligned}\cos\alpha &= \left(1 - \frac{r+r_0}{2a}\right) - \frac{c}{2a} = 1 - \frac{r+r_0+c}{2a} = 1 - \frac{s}{a}\\ \sin\beta &= \left(1 - \frac{r+r_0}{2a}\right) + \frac{c}{2a} = 1 - \frac{r+r_0-c}{2a} = 1 - \frac{s-c}{a}\end{aligned} \tag{10}$$

This part of the derivation makes use of the following three relationships:

$$\cos\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2} = 1 - \frac{r+r_0}{2}$$

$$\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2} = \sin\frac{E-E_0}{2}\sqrt{1-\left(e\cos\frac{E+E_0}{2}\right)^2}$$

$$\left(\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2}\right)^2 = \left(\frac{x-x_0}{2a}\right)^2 + \left(\frac{y-y_0}{2a}\right)^2 = \left(\frac{c}{2a}\right)^2$$

With the use of the half angle formulas given by

$$\sin\frac{\alpha}{2} = \sqrt{\frac{s}{2a}}$$
$$\sin\frac{\beta}{2} = \sqrt{\frac{s-c}{2a}}$$

and several additional substitutions, we have the *time-of-flight* form of Lambert's theorem

$$t = \sqrt{\frac{a^3}{\mu}}\left[(\alpha - \beta) - (\sin\alpha - \sin\beta)\right] \tag{11}$$

A discussion about the angles α and β can be found in "Geometrical Interpretation of the Angles α and β in Lambert's Problem" by J. E. Prussing, *AIAA Journal of Guidance and Control*, Volume 2, Number 5, Sept.-Oct. 1979, pages 442-443.

The algorithm used in this ***Numerit*** program is based on the method described in "A Practical Note on the Use of Lambert's Equation" by Geza Gedeon, *AIAA Journal*, Volume 3, Number 1, 1965, pages 149-150. This iterative solution is valid for elliptic, parabolic and hyperbolic transfer orbits which may be either posigrade or retrograde, and involve one or more revolutions about the central body. Additional information can also be found in G. S. Gedeon, "Lambertian Mechanics", *Proceedings of the 12th International Astronautical Congress*, Vol. I, 172-190.

The elliptic form of the general Lambert Theorem is

$$t = \sqrt{\frac{a^3}{\mu}}\left[(1-k)m\pi + k(\alpha - \sin\alpha) \mp (\beta - \sin\beta)\right] \tag{12}$$

where k may be either + 1 (posigrade) or −1 (retrograde), and m is the number of revolutions about the central body.

The Gedeon algorithm introduces the following variable

$$z = \frac{s}{2a}$$

and solves the problem with a Newton-Raphson iterative procedure. In this equation, a is the semimajor axis of the transfer orbit and

$$s = \frac{r_1 + r_2 + c}{2}$$

This algorithm also makes use of the following geometric constant

$$w = \pm\sqrt{1 - \frac{c}{s}}$$

The function to be solved iteratively is given by

$$N(z) = \frac{1}{z|z|^{1/2}2^{1/2}}\left\{\frac{1-k}{2}m\pi + k\left[|z|^{1/2} - |z|^{1/2}(1-z)^{1/2}\right] - \left[w|z|^{1/2}(1-w^2z)^{1/2}\right]\right\} \tag{13}$$

The Newton-Raphson algorithm also requires the derivative of this equation which is given by the following expression:

$$N'(z) = \frac{dN}{dz} = \frac{1}{|z|2^{1/2}}\left\{\frac{k}{(1-z)^{1/2}} - \frac{w^3}{(1-w^2z)^{1/2}} - \frac{3N(z)}{2^{1/2}}\right\} \tag{14}$$

The iteration for z is as follows:

$$z_{n+1} = z_n - \frac{N(z_n)}{N'(z_n)} \tag{15}$$

The syntax and function arguments for the ***Numerit*** function that solves this classic problem are as follows:

```
function lambfunc(ri, rf, tof, direct, revmax, statev, nsol)

` solve Lambert's orbital two point boundary value problem

` input

`  ri     = initial ECI position vector (kilometers)
`  rf     = final ECI position vector (kilometers)
`  tof    = time of flight (seconds)
`  direct = transfer direction (1 = posigrade, -1 = retrograde)
`  revmax = maximum number of complete orbits

` output

`  nsol   = number of solutions
`  statev = matrix of solutions
```

The Lambert inertial state vector and orbital elements at the *initial* time are returned in the two-dimensional `statev` array which is organized as follows:

```
statev(1, sn) = position vector x component
statev(2, sn) = position vector y component
statev(3, sn) = position vector z component
statev(4, sn) = velocity vector x component
statev(5, sn) = velocity vector y component
statev(6, sn) = velocity vector z component
statev(7, sn) = semimajor axis
statev(8, sn) = orbital eccentricity
statev(9, sn) = orbital inclination
statev(10, sn) = argument of perigee
statev(11, sn) = right ascension of the ascending node
statev(12, sn) = true anomaly
```

In this array `sn` is the solution number.

References and Bibliography

"Lunar Trajectories", NASA TN D-866, August 1961.

"Earth-Moon Trajectories", JPL Technical Report No. 32-503, May 1, 1964.

"Three-Dimensional Lunar Trajectories", V. A. Egorov, Mechanics of Space Flight Series, Israel

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"Circumlunar Trajectory Calculations", MIT Instrumentation Laboratory Report R-353, April 1962.

"Optimal Low Thrust Trajectories to the Moon", John T. Betts and Sven O. Erb, SIAM Journal on Applied Dynamical Systems, Vol. 2, No. 2, pp. 144-170, 2003.

"Integrated Algorithm for Lunar Transfer Trajectories Using a Pseudostate Technique", R. V. Ramanan, AIAA Journal of Guidance, Control and Dynamics, Vol. 25, No. 5, September-October 2002, pp. 946-952.

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"Injection Conditions for Lunar Trajectories", R. Kolenkiewicz and W. Putney, NASA TM X-55390, November 1965.

"Coplanar Three-Body Trans-Earth Lunar Trajectory Simulation Methodology", H. Ikawa, AIAA 88-0381, AIAA 26th Aerospace Sciences Meeting, Reno, Nevada, January 11-14, 1988.