Single Impulse De-orbit from Circular and Elliptical Orbits

This document summarizes equations that can be used to determine the characteristics of single impulse, coplanar two-body de-orbit trajectories from circular and elliptical orbits. For both types of initial orbits, the de-orbit maneuver consists of a tangential impulsive ΔV applied opposite to the velocity vector of the initial orbit at the maneuver orbital location. For de-orbit from elliptical orbits, the maneuver is performed at apogee. The user-defined mission constraints at the entry interface (EI) are altitude and flight path angle relative to a spherical Earth.

De-orbit from a circular orbit

The scalar magnitude of the single impulsive maneuver required to de-orbit a spacecraft from an initial circular orbit can be determined from the following expression

$$\Delta V = V_{c_e}\sqrt{\frac{1}{\tilde{r}}}\left\{1-\sqrt{\frac{2\left(\tilde{r}-1\right)}{\left(\dfrac{\tilde{r}}{\cos\gamma_e}\right)^2-1}}\right\} = V_{c_i}\left\{1-\sqrt{\frac{2\left(\tilde{r}-1\right)}{\left(\dfrac{\tilde{r}}{\cos\gamma_e}\right)^2-1}}\right\}$$

where

$\tilde{r} = \dfrac{h_i + r_{eq}}{h_e + r_{eq}} = \dfrac{r_i}{r_e} =$ radius ratio

$V_{c_e} = \sqrt{\dfrac{\mu}{\left(h_e + r_{eq}\right)}} = \sqrt{\dfrac{\mu}{r_e}} =$ local circular velocity at entry interface

$V_{c_i} = \sqrt{\dfrac{\mu}{\left(h_i + r_{eq}\right)}} = \sqrt{\dfrac{\mu}{r_i}} =$ local circular velocity of initial circular orbit

$\gamma_e =$ flight path angle at entry interface

$h_i =$ altitude of initial circular orbit

$h_e =$ altitude at entry interface

$r_i =$ radius of initial circular orbit

$r_e =$ radius at entry interface

$r_{eq} =$ Earth equatorial radius

$\mu =$ Earth gravitational constant

This algorithm is described in the technical article, “Deboost from Circular Orbits”, A. H. Milstead, *The Journal of the Astronautical Sciences*, Vol. XIII, No. 4, pp. 170-171, Jul-Aug., 1966. Additional information can be found in Chapter 5 of *Hypersonic and Planetary Entry Flight Mechanics* by Vinh, Busemann and Culp, The University of Michigan Press.

The true anomaly on the de-orbit trajectory at the entry interface θ_e can be determined from the following two equations:

$$\sin\theta_e = \frac{\dot{r}}{e_d}\sqrt{\frac{a_d\left(1-e_d^2\right)}{\mu}}$$

$$\cos\theta_e = \frac{a_d\left(1-e_d^2\right)}{e_d r_e} - \frac{1}{e_d}$$

and the following four quadrant inverse tangent operation

$$\theta_e = \tan^{-1}\left(\sin\theta_e, \cos\theta_e\right)$$

where

$e_d =$ eccentricity of the de-orbit trajectory

$a_d =$ semimajor axis of the de-orbit trajectory

$$\dot{r} = -\sqrt{\frac{\mu\left[2a_d r_e - r_e^2 - a_d^2\left(1-e_d^2\right)\right]}{a_d r_e^2}}$$

The elapsed time-of-flight between perigee of the de-orbit trajectory and the entry true anomaly θ_e is given by

$$t\left(\theta_e\right) = \frac{\tau}{2\pi}\left[2\tan^{-1}\left\{\sqrt{\frac{1-e_d}{1+e_d}}\tan\frac{\theta_e}{2}\right\} - \frac{e_d\sqrt{1-e_d^2}\sin\theta_e}{1+e_d\cos\theta_e}\right]$$

In this equation τ is the Keplerian orbital period of the de-orbit trajectory and is equal to $2\pi\sqrt{a_d^3/\mu}$.

Therefore, the flight time between the de-orbit impulse and entry interface is given by

$$\Delta t = t\left(\theta_e\right) - t\left(180^\circ\right) = t\left(\theta_e\right) - \frac{\tau}{2}$$

Finally, the orbital speed at the entry interface V_e can be determined from

$$V_e = \sqrt{\frac{2\mu}{r_e} - \frac{\mu}{a_d}}$$

Here’s a numerical example that uses these equations. For this example, the entry altitude is 121.92 kilometers and entry flight path angle is -2 degrees. The impulsive maneuver creates an elliptical orbit with an apogee altitude equal to the altitude of the initial circular orbit.

```
initial orbit

altitude                       400.000000  kilometers

entry altitude                 121.920000  kilometers

entry fpa                       -2.000000  degrees

entry trajectory

semimajor axis              6545.28443641  kilometers

eccentricity                   0.03557608

argument of perigee          180.00000000  degrees

perigee altitude             -65.71112719  kilometers

apogee altitude              400.00000000  kilometers

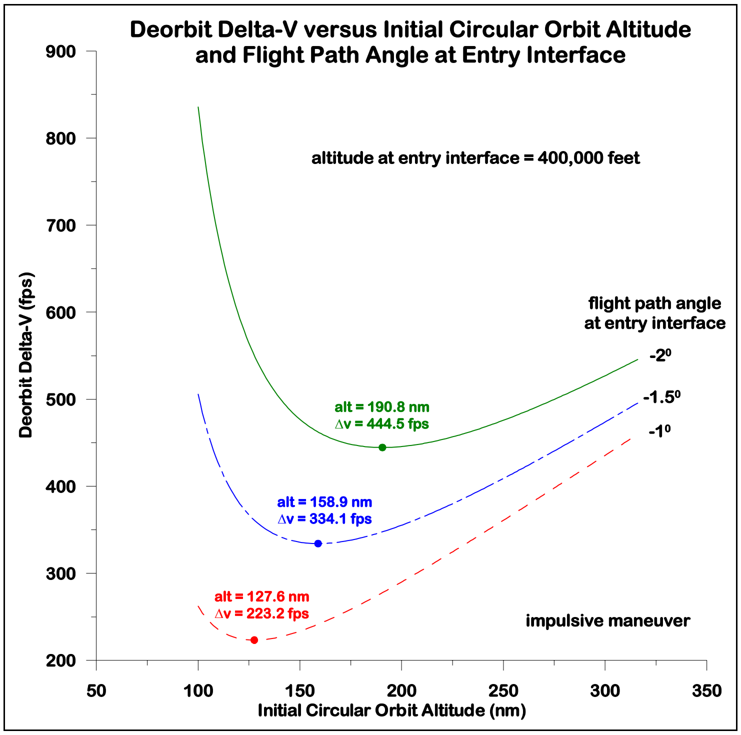
entry true anomaly           279.19205809  degrees

entry velocity              7857.88102977  meters/second

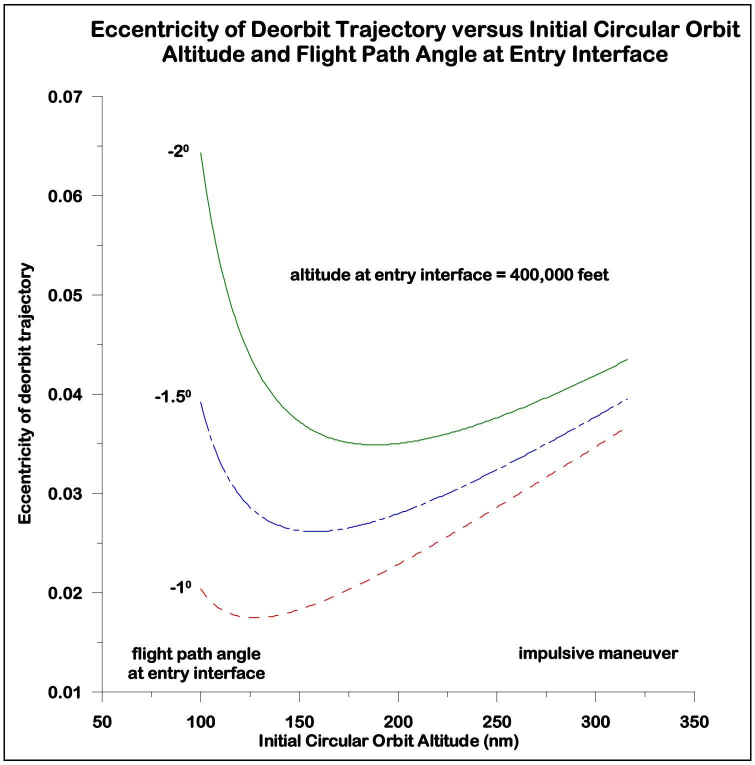
impulse-to-entry time         25.17812758  minutes

deorbit delta-v              137.64389361  meters/second
```

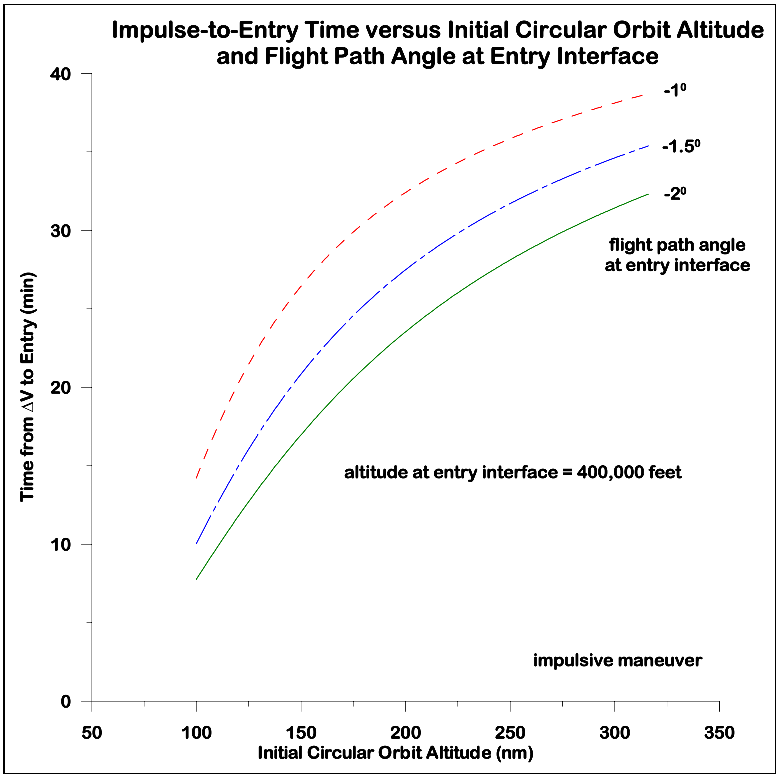
The following is a graphic display of the single impulsive ΔV required to de-orbit from a range of circular Earth orbit altitudes and several entry interface flight path angles. The entry interface altitude for this example is 400,000 feet.

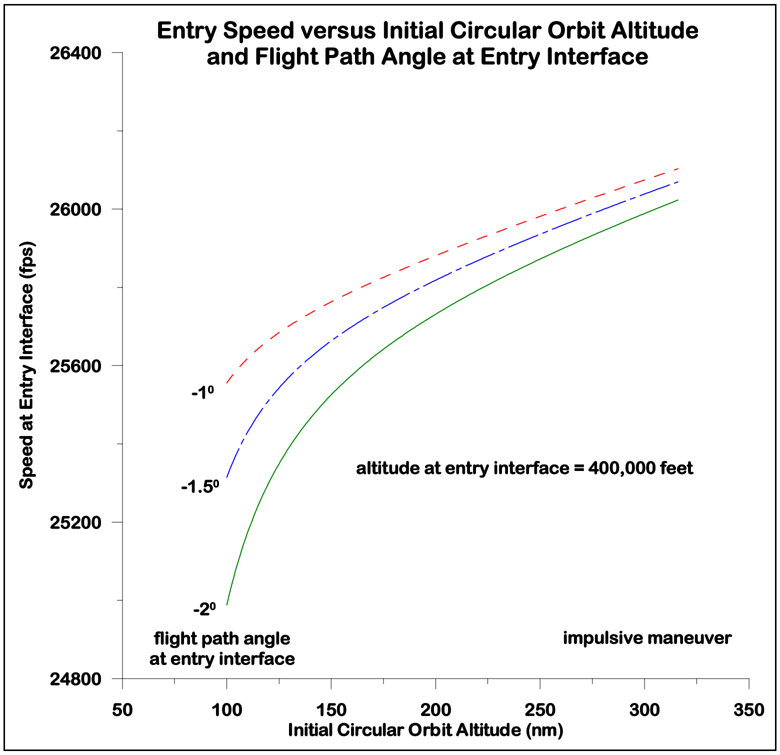


This plot is also labeled with the minimum de-orbit altitude and corresponding ΔV for a given flight path angle at the entry interface. An examination of the following plot of the eccentricity of the de-orbit trajectory reveals why these minima exist.



The next two plots illustrate the behavior of the impulse-to-entry mission time and speed at the entry interface.





De-orbit from an elliptical orbit

The scalar magnitude of the impulsive delta-v for de-orbit from an initial elliptical orbit is given by

$$\Delta V = \sqrt{\frac{\mu}{r_e}}\left(\sqrt{\frac{2\tilde{r}_p}{\tilde{r}_a\left(\tilde{r}_a+\tilde{r}_p\right)}} - \sqrt{\frac{2\left(\tilde{r}_a-1\right)}{\tilde{r}_a\left(\tilde{r}_a^2-\cos^2\gamma_e\right)}}\cos\gamma_e\right)$$

where

$r_e =$ geocentric radius at the entry altitude

$\tilde{r}_a = r_a / r_e$

$\tilde{r}_p = r_p / r_e$

$\gamma_e =$ flight path angle at entry

$r_a =$ apogee radius of the initial elliptical orbit

$r_p =$ perigee radius of the initial elliptical orbit

$\mu =$ gravitational constant of the Earth

The true anomaly at entry can be determined from the following series of equations.

$$\sin\theta_e = \frac{\dot{r}}{e_d}\sqrt{\frac{a_d\left(1-e_d^2\right)}{\mu}}$$

$$\cos\theta_e = \frac{a_d\left(1-e_d^2\right)}{e_d r_e} - \frac{1}{e_d}$$

$$\theta_e = \tan^{-1}\left(\sin\theta_e, \cos\theta_e\right)$$

where

$e_d =$ eccentricity of the de-orbit trajectory

$a_d =$ semimajor axis of the de-orbit trajectory

$$\dot{r} = -\sqrt{\frac{\mu\left[2a_d r_e - r_e^2 - a_d^2\left(1-e_d^2\right)\right]}{a_d r_e^2}}$$

The time-of-flight between perigee and the entry true anomaly θ_e is given by

$$t(\theta_e) = \frac{\tau}{2\pi}\left[2\tan^{-1}\left\{\sqrt{\frac{1-e_d}{1+e_d}}\tan\frac{\theta_e}{2}\right\} - \frac{e_d\sqrt{1-e_d^2}\sin\theta_e}{1+e_d\cos\theta_e}\right]$$

In this equation τ is the orbital period of the de-orbit trajectory.

Therefore, the flight time between the de-orbit impulse time and entry is given by

$$\Delta t = t(\theta_e) - t\left(180^\circ\right) = t(\theta_e) - \frac{\tau}{2}$$

Finally, the speed at reentry V_e can be determined from

$$V_e = \sqrt{\frac{2\mu}{r_e} - \frac{\mu}{a_d}}$$

Here’s a numerical example that uses these equations. For this example, the entry altitude is 111.252 kilometers and entry flight path angle is -4 degrees.

```
initial orbit

perigee altitude               285.798000  kilometers

apogee altitude              35785.922000  kilometers

semimajor axis               24414.000000  kilometers

eccentricity                   0.727044

entry altitude                 111.252000  kilometers

entry fpa                       -4.000000  degrees


de-orbit trajectory

semimajor axis             24308.08290588  kilometers

eccentricity                   0.73456961

perigee altitude              73.96381175  kilometers

apogee altitude            35785.92200000  kilometers

entry true anomaly           350.55084585  degrees

entry velocity             10317.40933180  meters/second
```

```
entry fpa                     -4.00000000  degrees

impulse-to-entry time        312.58844372  minutes

de-orbit delta-v              22.29796787  meters/second
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