FLIGHT PATH EQUATIONS OF MOTION

(Rotating Oblate Earth and Zonal Gravity Model)

The first-order flight path equations of motion relative to a rotating oblate Earth and a fourth-order zonal gravity model are as follows:

Geocentric radius

$$\dot{r} = \frac{dr}{dt} = V \sin\gamma$$

Geographic longitude

$$\dot{\lambda} = \frac{d\lambda}{dt} = V \frac{\cos\gamma \sin\psi}{r\cos\delta}$$

Geocentric declination

$$\dot{\delta} = \frac{d\delta}{dt} = V \frac{\cos\gamma \cos\psi}{r}$$

Speed

$$\dot{V} = \frac{dV}{dt} = \frac{\left(T\cos\alpha - D\right)}{m} - g_r \sin\gamma + g_\phi \cos\gamma \cos\psi$$
$$+ \omega_e^2 r \cos\delta \left(\sin\gamma \cos\delta - \sin\delta \cos\gamma \cos\psi\right)$$

Flight path angle

$$\dot{\gamma} = \frac{d\gamma}{dt} = \frac{V}{r}\cos\gamma + \left(\frac{T\sin\alpha + L}{mV}\right)\cos\beta - \frac{g_r \cos\gamma}{V} - \frac{g_\phi \sin\gamma \cos\psi}{V}$$
$$+2\omega_e \sin\psi \cos\delta + \omega_e^2 \frac{r}{V}\cos\delta\left(\cos\psi \sin\gamma \sin\delta + \cos\gamma \cos\delta\right)$$

Flight azimuth

$$\dot{\psi} = \frac{d\psi}{dt} = \frac{V}{r}\tan\delta \sin\psi \cos\gamma + \left(\frac{T\sin\alpha + L}{mV\cos\gamma}\right)\sin\beta - \frac{\sin\psi}{V\cos\gamma} g_\phi$$
$$+2\omega_e\left(\sin\delta - \cos\psi \cos\delta \tan\gamma\right) + \frac{r}{V\cos\gamma}\omega_e^2 \sin\psi \cos\delta \sin\delta$$

where

$r =$ geocentric radius
$v =$ speed
$\gamma =$ flight path angle
$\delta =$ geocentric declination
$\lambda =$ longitude $(+\text{ east})$
$\psi =$ flight azimuth $(+\text{ clockwise from north})$
$\beta =$ bank angle $(+\text{ for a right turn})$
$\alpha =$ angle of attack
$r_e =$ Earth equatorial radius
$\omega_e =$ Earth inertial rotation rate
$\mu =$ Earth gravitational constant
$L =$ aerodynamic lift force $= \frac{1}{2}\rho v^2 C_L S$
$D =$ aerodynamic drag force $= \frac{1}{2}\rho v^2 C_D S$
$T =$ propulsive thrust
$m =$ spacecraft mass
$C_L =$ lift coefficient (non-dimensional)
$C_D =$ drag coefficient (non-dimensional)
$S =$ aerodynamic reference area
$\rho =$ atmospheric density $= f(h)$
$h =$ geodetic altitude

The bank angle is the angle between the lift vector and the projection of the lift vector on the plane formed by the vehicle's relative velocity vector and the local vertical direction. Bank angle is measured positive clockwise looking forward in the direction of motion.

The components of the gravity vector are determined from the gradient of the potential function according to

$$\mathbf{F}_G = \nabla U = \begin{Bmatrix} \dfrac{1}{r}\dfrac{\partial U}{\partial \phi} \\ 0 \\ -\dfrac{\partial U}{\partial r} \end{Bmatrix} = \begin{Bmatrix} g_\phi \\ 0 \\ g_r \end{Bmatrix}$$

where

$$U = \frac{\mu}{r}\left[1 - \sum_{j=1}^{\infty}\left(\frac{r_e}{r}\right)^j J_j P_{j0}\left(\sin\phi\right)\right]$$

$$\frac{\partial U}{\partial r} = \frac{\mu}{r^2}\left[-1 + \sum_{l=2}^{\infty}\left(l+1\right)\left(\frac{r_e}{r}\right)^l J_l P_{l0}\left(\sin\phi\right)\right]$$

$$\frac{1}{r}\frac{\partial U}{\partial \phi} = -\frac{\mu}{r^2}\sum_{l=2}^{\infty}\left(\frac{r_e}{r}\right)^l J_l \frac{\partial P_{l0}}{\partial \phi}$$

$$P_{j0}\left(\sin\phi\right) = \sum_{t=0}^{k} T_{jt} \sin^{j-2t}\phi$$

$$T_{jt} = \left(-1\right)^t \left(2j-2t\right)!/\,2lt!\left(j-t\right)!\left(j-2t\right)!$$

For a *zonal-only* gravity model of order four, the Legendre functions and their partial derivatives are given by

$$P_{20} = \frac{1}{2}\left(3\sin^2\phi - 1\right)$$

$$P_{30} = \frac{1}{2}\left(5\sin^3\phi - 3\sin\phi\right)$$

$$P_{40} = \frac{1}{8}\left(35\sin^4\phi - 30\sin^2\phi + 3\right)$$

$$\frac{\partial P_{20}}{\partial \phi} = 3\sin\phi\cos\phi$$

$$\frac{\partial P_{30}}{\partial \phi} = \frac{3}{2}\left(5\sin^2\phi - 1\right)\cos\phi$$

$$\frac{\partial P_{40}}{\partial \phi} = \frac{5}{2}\left(7\sin^2\phi - 3\right)\sin\phi\cos\phi$$

Load Factors

"total" (wind axis coordinate system)

$$n = \frac{\sqrt{(L + T\sin\alpha)^2 + (D - T\cos\alpha)^2}}{W}$$

axial component (body frame coordinate system)

$$n = \frac{-L\sin\alpha + D\cos\alpha - T}{W}$$

normal component (body frame coordinate system)

$$n = \frac{L\cos\alpha + D\sin\alpha}{W}$$

Axial and Normal Coefficients (body axis coordinate system)

$$C_A = -C_L \sin\alpha + C_D \cos\alpha - T$$

$$C_N = C_L \cos\alpha + C_D \sin\alpha$$

Lift and Drag Coefficients (body axis coordinate system)

$$C_L = -C_A \sin\alpha + C_N \cos\alpha$$

$$C_D = C_A \cos\alpha + C_N \sin\alpha$$

Axial and Normal Force Components (body axis coordinate system)

$$\mathbf{f} = \frac{1}{2}\rho V^2 S \begin{Bmatrix} -C_A \\ 0 \\ -C_N \end{Bmatrix}$$

Steady-State Flight Conditions

$$T\cos\alpha - D - W\sin\gamma = 0$$

$$T\sin\alpha + L - W\cos\gamma = 0$$

Aerodynamic Characteristics

General Form of Drag Polar

$$C_D = C_{D_0} + k\left|C_L\right|^n$$

Lift-to-Drag Ratio

$$E = \frac{L}{D} = \frac{C_L}{C_D} = \frac{C_L}{C_{D_0} + kC_L^n}$$

Maximum Lift-to-Drag Ratio

$$E^* = \frac{dE}{dC_L} = \frac{\left(C_{D_0} + kC_L^n\right) - C_L\left(nkC_L^{n-1}\right)}{\left(C_{D_0} + kC_L^n\right)^2} = 0$$

$$C_L^* = \sqrt[n]{\frac{C_{D_0}}{k\left(n-1\right)}}$$

$$C_D^* = \frac{nC_{D_0}}{\left(n-1\right)}$$

In general,

$$E^* = \frac{C_L^*}{C_D^*} = \frac{\sqrt[n]{\left(n-1\right)^{n-1}}}{n\sqrt[n]{kC_{D_0}^{n-1}}}$$

For a parabolic drag polar $\left(n = 2\right)$,

$$C_D = C_{D_0} + kC_L^2$$

where

$C_D =$ drag coefficient

$C_{D_0} =$ drag coefficient at 0° angle-of-attack

$C_L =$ lift coefficient

$k =$ constant

and

$$C_D^* = 2C_{D_0} = \text{ drag coefficient at maximum } \mathrm{L/D}$$

$$C_L^* = \sqrt{C_{D_0} / k} = \text{ lift coefficient at maximum } \mathrm{L/D}$$

$$E^* = \left(\frac{C_L}{C_D}\right)_{\max} = \frac{1}{2\sqrt{kC_{D_0}}} = \text{ maximum } \mathrm{L/D}$$

Chapman’s Stagnation Point Heat Rate Equation

$$\dot{q} = \frac{dq}{dt} = \frac{17,600}{\sqrt{R_N}}\left(\frac{v}{v_0}\right)^{3.15}\sqrt{\frac{\rho}{\rho_0}} \qquad \left(\frac{\text{BTU}}{\text{ft}^2 - \text{sec}}\right)$$

where

$R_N =$ nose radius (feet)

$v =$ relative velocity at the spacecraft location (feet/second)

$v_0 =$ "local circular velocity" at the Earth's surface $= \sqrt{\frac{\mu}{r_e}}$ (feet/second)

$\rho =$ atmospheric density at the spacecraft location (slugs/feet^3)

$\rho_0 =$ atmospheric density at the Earth's surface (slugs/feet^3)

$\mu =$ gravitational constant of the Earth (feet^3/second^2)

$r_e =$ radius of the Earth (feet)

Crossrange and Downrange Calculations (Spherical Earth)

The crossrange angle is determined from the following expression:

$$\sin \nu = -\sin \psi_1 \sin \phi_2 \cos \phi_1 \cos \Delta\lambda - \cos \psi_1 \cos \phi_1 \sin \Delta\lambda + \sin \psi_1 \cos \phi_2 \sin \phi_1$$

The downrange angle is determined from the following three equations:

$$\sin \mu = -\cos \psi_1 \sin \phi_2 \cos \phi_1 \cos \Delta\lambda + \sin \psi_1 \cos \phi_1 \sin \Delta\lambda + \cos \psi_1 \cos \phi_2 \sin \phi_1$$

$$\cos \mu = \cos \phi_2 \cos \phi_1 \cos \Delta\lambda + \sin \phi_2 \sin \phi_1$$

$$\mu = \tan^{-1} \left(\sin \mu, \cos \mu \right)$$

where

$\phi_1 =$ geocentric latitude of the initial point

$\psi_1 =$ flight azimuth at the initial point

$\phi_2 =$ geocentric latitude of the final point

$\Delta\lambda = \lambda_2 - \lambda_1$

$\lambda_1 =$ east longitude of the initial point

$\lambda_2 =$ east longitude of the final point

The crossrange distance d_c and downrange distance d_d are determined from

$$d_c = r_e \nu$$

$$d_d = r_e \mu$$

where r_e is the radius of the Earth. The flight azimuth is measured positive clockwise from north. Also note that the inverse tangent above is a four quadrant form.

UTILITY TRANSFORMATIONS

Conversion of ECI state vector to spherical (ADBARV) coordinates

The components of the ADBARV coordinate system are as follows:

Alpha = right ascension

Delta = geocentric declination

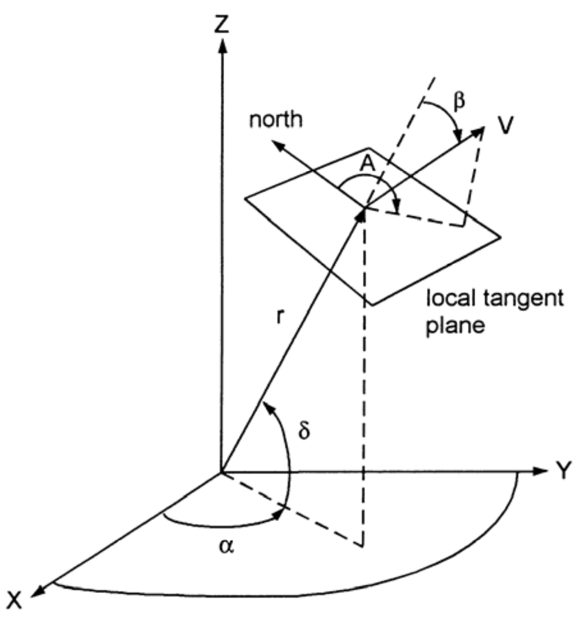
Beta = conjugate flight path angle

A = azimuth

R = position magnitude

V = velocity magnitude

The following diagram illustrates the geometry of the ADBARV coordinates. In this picture α is the right ascension, δ is the geocentric declination and β is the conjugate flight path angle.



The mathematical relationships between ADBARV elements and the components of the ECI position and velocity vectors are as follows:

$$r = \sqrt{r_x^2 + r_y^2 + r_z^2}$$

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

$$\alpha = \tan^{-1}\left(r_y, r_x\right)$$

$$\delta = \tan^{-1}\left(r_z, \sqrt{r_x^2 + r_y^2}\right)$$

$$\beta = \cos^{-1}\left(\frac{\mathbf{r} \bullet \mathbf{v}}{\left|\mathbf{r} \bullet \mathbf{v}\right|}\right)$$

$$A = \tan^{-1}\left[r\left(r_x v_y - r_y v_x\right), r_y\left(r_y v_z - r_z v_y\right) - r_x\left(r_z v_x - r_x v_z\right)\right]$$

Conversion of spherical (ADBARV) coordinates to ECI state vector

The inertial position and velocity vectors can be determined from the ADBARV elements with the following set of equations:

$$r_x = r\cos\delta\cos\alpha$$

$$r_y = r\cos\delta\sin\alpha$$

$$r_z = r\sin\delta$$

$$v_x = v\left[\cos\alpha\left(-\cos A\sin\beta\sin\delta + \cos\beta\cos\delta\right) - \sin A\sin\beta\sin\alpha\right]$$

$$v_y = v\left[\sin\alpha\left(-\cos A\sin\beta\sin\delta + \cos\beta\cos\delta\right) + \sin A\sin\beta\cos\alpha\right]$$

$$v_z = v\left(\cos A\cos\delta\sin\beta + \cos\beta\cos\delta\right)$$

The inertial speed can also be computed from the following expression

$$v_i = \sqrt{v^2 + 2v\,r\omega\cos\gamma\sin\psi\cos\delta + r^2\omega^2\cos^2\delta}$$

The inertial flight path angle can be computed from

$$\cos\gamma_i = \sqrt{\frac{v^2\cos^2\gamma + 2vr\omega\cos\gamma\cos\psi\cos\delta + r^2\omega^2\cos^2\delta}{v^2 + 2vr\omega\cos\gamma\cos\psi\cos\delta + r^2\omega^2\cos^2\delta}}$$

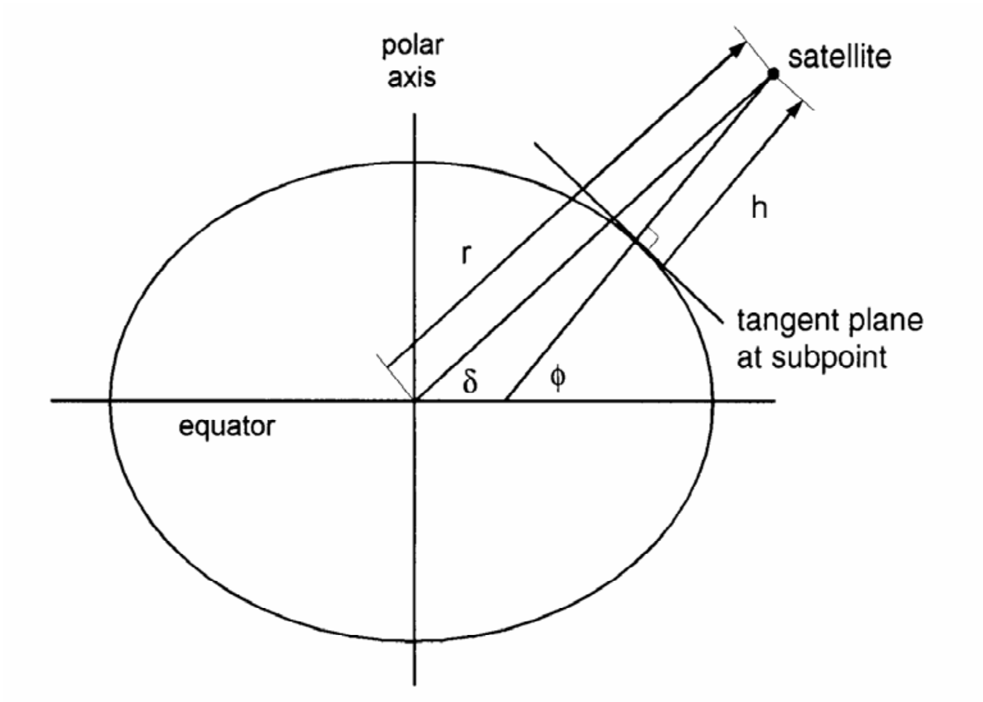
The inertial azimuth can be computed from

$$\cos\psi_i = \frac{v\cos\gamma\cos\psi + r\omega\cos\delta}{\sqrt{v^2\cos^2\gamma + 2vr\omega\cos\gamma\cos\psi\cos\delta + r^2\omega^2\cos^2\delta}}$$

where all coordinates on the right-hand-side of these equations are relative to a rotating Earth.

Conversion of geocentric radius and declination to geodetic altitude and latitude

The following diagram illustrates the geometric relationship between geocentric and geodetic coordinates for an oblate spheroid.



In this diagram, δ is the *geocentric* declination, ϕ is the *geodetic* latitude, *r* is the geocentric radius, and *h* is the geodetic altitude. The exact mathematical relationship between geocentric and geodetic coordinates is given by the following system of two nonlinear equations

$$(c+h)\cos\phi - r\cos\delta = 0$$

$$(s+h)\sin\phi - r\sin\delta = 0$$

where the geodetic constants *c* and *s* are given by

$$c = \frac{r_{eq}}{\sqrt{1-\left(2f - f^2\right)\sin^2\varphi}}$$

$$s = c\left(1-f\right)^2$$

and r_{eq} is the Earth equatorial radius (6378.14 kilometers) and f is the flattening factor for the Earth (1/298.257).

The geodetic latitude is determined using the following expression:

$$\phi = \delta + \left(\frac{\sin 2\delta}{\rho}\right) f + \left[\left(\frac{1}{\rho^2} - \frac{1}{4\rho}\right)\sin 4\delta\right] f^2$$

The geodetic altitude is calculated from

$$\hat{h} = \left(\hat{r} - 1\right) + \left\{\left(\frac{1 - \cos 2\delta}{2}\right) f + \left[\left(\frac{1}{4\rho} - \frac{1}{16}\right)\left(1 - \cos 4\delta\right)\right] f^2\right\}$$

In these equations, ρ is the geocentric distance of the vehicle, $\hat{h} = h / r_{eq}$ and $\hat{r} = \rho / r_{eq}$.

Conversion of geodetic latitude and altitude to geocentric radius and geocentric declination

The equations for this coordinate conversion are as follows:

$$\delta = \phi + \left(\frac{-\sin 2\phi}{\hat{h}+1}\right) f + \left\{\frac{-\sin 2\phi}{2\left(\hat{h}+1\right)^2} + \left[\frac{1}{4\left(\hat{h}+1\right)^2} + \frac{1}{4\left(\hat{h}+1\right)}\right]\sin 4\phi\right\} f^2$$

and

$$\hat{\rho} = \left(\hat{h}+1\right) + \left(\frac{\cos 2\phi - 1}{2}\right) f + \left\{\left[\frac{1}{4\left(\hat{h}+1\right)} + \frac{1}{16}\right]\left(1 - \cos 4\phi\right)\right\} f^2$$

where the geocentric radius r and geodetic altitude h have been normalized by $\hat{\rho} = r / r_{eq}$ and $\hat{h} = h / r_{eq}$, respectively, and r_{eq} is the equatorial radius of the Earth.