Mission Constraints and Trajectory Optimization

This memo summarizes the mathematical and geometric relationships that are often used in trajectory optimization computer programs. In order to avoid singularities due to trajectory type and "wrapping" of angular orbital elements during iterative computations, it is preferable to use one or more individual components of the state vector as well as quantities derived from components of the state vector (angular momentum, energy, trajectory eccentricity, etc.). These elements are usually called *mission constraints* or *targeting parameters*. Within a trajectory optimization program, subsets of these constraints may be enforced at intermediate trajectory points and at park or mission orbit injection in order to satisfy a variety of mission requirements.

The following table illustrates the relationship between classical orbital elements and the corresponding mission constraint subsets. In this table, a is semimajor axis, e is orbital eccentricity, i is orbital inclination, ω is argument of periapsis, and Ω is the right ascension of the ascending node. Bold letters and symbols indicate vectors.

Table 1. Orbital Elements and Mission Constraint Subsets

orbital element	**constraint subset**						
	$\mathbf{h}, r, \gamma$	$\mathbf{h}, r, \gamma$	h, h_z, r, γ	h, h_z, C_3	h, h_z, C_3, e_z	$\mathbf{h}, C_3$	$\mathbf{h}, \mathbf{e}$
a	constrained	constrained	constrained	constrained	constrained	constrained	constrained
e	$= 0$	$= 0$	$= 0$	> 0	> 0	> 0	> 0
i	$= 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	constrained	constrained
ω	undefined	undefined	undefined	free	constrained	free	constrained
Ω	undefined	constrained	free	free	free	constrained *	constrained *

*undefined if $i = 0$

where

$\mathbf{h} =$ angular momentum vector

$\mathbf{e} =$ orbital eccentricity vector

$r =$ geocentric radius

$\gamma =$ flight path angle

$C_3 =$ specific orbital energy

$h =$ angular momentum magnitude

$h_z =$ z-component of angular momentum vector

$e_z =$ z-component of eccentricity vector

This next table illustrates example constraint subsets for each classical orbital element. Constraints for special cases such as circular or equatorial orbits are also noted.

Table 2. Example Mission Constraints

constraint elements	equivalent constraint
C_3 or r and v (circular orbit)	a
h and C_3 or r, v and γ (circular orbit)	e
h and h_z or $\mathbf{h}$ $(i = 0)$	i
h, h_z, e_z and C_3	ω
h_x and h_y	Ω

Mission Constraint Equations

This section summarizes the equations used to compute the complete set of mission constraint components from the inertial state vector. In the equations that follow, $\mathbf{r}$ is the inertial position vector and $\mathbf{v}$ is the inertial velocity vector of the spacecraft. A "hat" symbol indicates a unit vector and μ is the gravitational constant of the central gravitational body.

The angular momentum vector is normal to the orbit plane and is given by the following right-handed cross product

$$\mathbf{h} = \mathbf{r} \times \mathbf{v}$$

The unit angular momentum vector is given by

$$\hat{\mathbf{h}} = \frac{\mathbf{r} \times \mathbf{v}}{|\mathbf{r} \times \mathbf{v}|}$$

The angular momentum magnitude can be computed from

$$h = |\mathbf{r} \times \mathbf{v}| = \sqrt{h_x^2 + h_y^2 + h_z^2} = \sqrt{\mu a \left(1 - e^2\right)}$$

The flight path angle can be determined from

$$\gamma = \sin^{-1}\left(\frac{\mathbf{r} \bullet \mathbf{v}}{|\mathbf{r} \bullet \mathbf{v}|}\right) = \sin^{-1}\left(\hat{\mathbf{r}} \bullet \hat{\mathbf{v}}\right)$$

Trajectory computer programs will often enforce the *sine* of the flight path angle within targeting algorithms. This approach avoids a trigonometric calculation and a numerical ambiguity since the flight angle takes on values between ± 90 degrees $\left(-1 \le \sin\gamma \le +1\right)$. Flight path angle is positive above the local horizon and negative below.

For targeting to final circular orbits, the following constraint equation is also useful

$$\frac{\mathbf{r} \bullet \mathbf{v}}{\sqrt{\mu r}} = 0$$

Twice the specific (per unit mass) orbital energy can be determined from

$$C_3 = v^2 - \frac{2\mu}{r} = -\frac{\mu}{a}$$

The unit eccentricity vector is in the direction of perigee and is given by

$$\hat{\mathbf{e}} = \frac{\mathbf{v} \times \mathbf{h}}{\mu} - \hat{\mathbf{r}} = \frac{1}{\mu}\left[\left(v^2 - \frac{\mu}{r}\right)\mathbf{r} - (\mathbf{r} \bullet \mathbf{v})\,\mathbf{v}\right]$$

Orbital Element Equations

This section describes the equations used to recover the classical *angular* or *orientation* orbital elements from derived vector elements. Several useful mission constraints are also discussed in this section. The correct quadrant for each angular orbital element can be determined from the corresponding sine and cosine equations using a four quadrant inverse tangent evaluation.

The semimajor axis can be determined from one of the following equations:

$$a = \frac{|\mathbf{r}|}{2 - |\mathbf{r}||\mathbf{v}|^2 / \mu} = -\frac{\mu h^2}{\left\{(e\mu)^2 - \mu^2\right\}} = \frac{\mu r}{\left(2\mu - r v^2\right)} = -\frac{\mu}{C_3}$$

The sine and cosine of orbital inclination can be determined from the following two expressions:

$$\sin i = \left|\hat{\mathbf{k}} \times \hat{\mathbf{h}}\right|$$

$$\cos i = \hat{\mathbf{h}} \bullet \hat{\mathbf{k}} = \frac{h_z}{h} = \hat{h}_z$$

where $\hat{\mathbf{k}}$ is a geocentric unit vector in the direction of the planet's spin axis and is given by $[0\;0\;1]^T$. An orbital inclination constraint can be enforced using the z-component and magnitude of the angular momentum vector according to the second equation.

A geocentric unit vector in the direction of the ascending node is given by the following cross product

$$\hat{\mathbf{n}} = \hat{\mathbf{k}} \times \hat{\mathbf{h}}$$

The sine and cosine of the right ascension of the ascending node (RAAN) can be determined from

$$\sin\Omega = \left(\hat{\mathbf{i}} \times \hat{\mathbf{n}}\right) \bullet \hat{\mathbf{k}}$$

$$\cos\Omega = \hat{\mathbf{n}} \times \hat{\mathbf{i}}$$

where $\hat{\mathbf{i}}$ is a geocentric unit vector in the direction of the x-axis of the planet-centered-inertial (ECI) coordinate system and is given by $\left[1\ 0\ 0\right]^T$.

A RAAN constraint can be enforced using the x and y components of the angular momentum vector according to the expression

$$\Omega = \tan^{-1}\left(h_x, -h_y\right)$$

The sine and cosine of the argument of periapsis for elliptical and hyperbolic orbits can be determined from the next two expressions

$$\sin\omega = \left(\hat{\mathbf{n}} \times \hat{\mathbf{e}}\right) \bullet \hat{\mathbf{h}}$$

$$\cos\omega = \hat{\mathbf{h}} \bullet \hat{\mathbf{e}}$$

A convenient constraint equation for the argument of perigee is given by

$$\omega = \tan^{-1}\left(e_z, e\, g_z\right)$$

where $\mathbf{g} = \mathbf{h} \times \mathbf{e} / h\, e$ and the evaluation requires a four quadrant inverse tangent.

A useful expression for the scalar orbital eccentricity in terms of specific orbital energy and angular momentum is given by

$$e = \frac{h\sqrt{C_3 + \mu^2 / h^2}}{\mu}$$

In these equations, the dual argument inverse tangent function is a four quadrant operation.

Targeting with Modified Equinoctial Orbital Elements

The modified equinoctial orbital elements are a set of orbital elements that are useful for trajectory analysis and optimization. They are valid for circular, elliptic, and hyperbolic orbits. These *direct* modified equinoctial equations exhibit no singularity for zero eccentricity and orbital inclinations equal to 0 and 90 degrees. However, please note that two of the components are singular for an orbital inclination of 180 degrees.

The classic reference for these elements is “A Set of Modified Equinoctial Orbital Elements”, M. J. H. Walker, B. Ireland and J. Owens, *Celestial Mechanics*, Vol. 36, pp. 409-419, 1985.

The modified equinoctial elements are defined in terms of the classical orbital elements as follows:

$$p = a\left(1 - e^2\right)$$

$$f = e\cos\left(\omega + \Omega\right)$$

$$g = e\sin\left(\omega + \Omega\right)$$

$$h = \tan\left(i/2\right)\cos\Omega$$

$$k = \tan\left(i/2\right)\sin\Omega$$

$$L = \theta + \omega + \Omega$$

where

$p =$ semiparameter
$a =$ semimajor axis
$e =$ orbital eccentricity
$i =$ orbital inclination
$\omega =$ argument of perigee
$\Omega =$ right ascension of the ascending node
$\theta =$ true anomaly
$L =$ true longitude

The classical orbital elements can be recovered from the modified equinoctial orbital elements with

semimajor axis

$$a = \frac{p}{1 - f^2 - g^2}$$

orbital eccentricity

$$e = \sqrt{f^2 + g^2}$$

orbital inclination

$$i = 2\tan^{-1}\left(\sqrt{h^2 + k^2}\right)$$

argument of periapsis

$$\omega = \tan^{-1}\left(g, f\right) - \tan^{-1}\left(k, h\right)$$

$$\sin\omega = \frac{g\,h - f\,k}{e\tan\left(i/2\right)}$$

$$\cos\omega = \frac{f\,h + g\,k}{e\tan\left(i/2\right)}$$

right ascension of the ascending node

$$\Omega = \tan^{-1}\left(k, h\right)$$

$$\sin\Omega = \frac{k}{\tan\left(i/2\right)}$$

$$\cos\Omega = \frac{h}{\tan\left(i/2\right)}$$

true anomaly

$$\theta = L - \left(\omega + \Omega\right) = L - \tan^{-1}\left(g, f\right)$$

$$\sin\theta = \frac{1}{e}\left(f\sin L - g\cos L\right)$$

$$\cos\theta = \frac{1}{e}\left(f\cos L + g\sin L\right)$$

In these expressions, an inverse tangent expression of the form $\theta = \tan^{-1}\left(a, b\right)$ denotes a four quadrant evaluation where $a = \sin\theta$ and $b = \cos\theta$.

Constraint formulations that enforce both the sine and cosine of a desired orbital element should be used whenever possible. This approach involves a combination of equality and inequality constraints and ensures that the "targeted" orbital element is in the correct quadrant.

To illustrate this technique, here are several examples for different values of argument of perigee and the corresponding mission constraints:

$$0^\circ < \omega < 90^\circ \rightarrow \begin{cases} \sin\omega > 0 \rightarrow g\,h - f\,k > 0 \\ f\,h + g\,k = e\tan\left(i/2\right)\cos\omega \end{cases}$$

$$\omega = 270^\circ \rightarrow \begin{cases} \sin\omega \le 0 \rightarrow g\,h - f\,k \le 0 \\ \cos\omega = 0 \rightarrow f\,h + g\,k = 0 \end{cases}$$

$$\omega = 178^\circ \rightarrow \begin{cases} g\,h - f\,k = e\tan\left(i/2\right)\sin\omega \\ \cos\omega \le 0 \rightarrow f\,h + g\,k \le 0 \end{cases}$$

The following is a *sign* table of the sine and cosine for each quadrant.

quadrant	sine	cosine
1	+	+
2	+	−
3	−	−
4	−	+

orbital eccentricity constraint

$$e = \sqrt{f^2 + g^2}$$

For a <u>circular</u> orbit, $f = g = 0$.

orbital inclination constraint

$$\tan\left(\frac{i}{2}\right) = \sqrt{h^2 + k^2}$$

For an <u>equatorial</u> orbit, $h = k = 0$.

argument of perigee constraints

$$g\,h - f\,k = e\sin\omega\tan\left(i/2\right) \to \sin\omega = \frac{g\,h - f\,k}{e\tan\left(i/2\right)}$$

$$f\,h + g\,k = e\cos\omega\tan\left(i/2\right) \to \cos\omega = \frac{f\,h + g\,k}{e\tan\left(i/2\right)}$$

right ascension of the ascending node constraints

$$k = \tan\left(i/2\right)\sin\Omega \to \sin\Omega = \frac{k}{\tan\left(i/2\right)}$$

$$h = \tan\left(i/2\right)\cos\Omega \to \cos\Omega = \frac{h}{\tan\left(i/2\right)}$$

true anomaly constraints

$$\theta = L - \left(\omega + \Omega\right) = L - \tan^{-1}\left(g, f\right)$$

In general,

$$\sin\theta = \frac{1}{e}\left(f \sin L - g \cos L \right)$$

$$\cos\theta = \frac{1}{e}\left(f \cos L + g \sin L \right)$$

For a circular orbit,

$$\sin\theta = \sin L \cos\Omega - \cos L \sin\Omega$$

$$\cos\theta = \cos L \cos\Omega + \sin L \sin\Omega$$

For a circular, equatorial orbit,

$$\theta = L\,,\ \sin\theta = \sin L \ \text{ and } \ \cos\theta = \cos L\ .$$

Targeting Example

For a user-defined semimajor axis, eccentricity and inclination, the set of modified equinoctial constraints are as follows:

$$p = \tilde{p}$$

$$\sqrt{f^2 + g^2} = \tilde{e}$$

$$\sqrt{h^2 + k^2} = \tan\left(\tilde{i}/2\right)$$

where the tilde indicates the value of the user-defined classical orbital element.

Targeting to an Outgoing Hyperbola

For interplanetary missions, the "target specs" are usually specified by the specific orbital energy C_3, and the right ascension (RLA) and declination (DLA) of the outgoing asymptote of the launch or departure hyperbola. This section describes the computation of an "energy-scaled" mission constraint useful for these types of missions.

A unit vector in the direction of the departure asymptote is given by

$$\hat{\mathbf{S}} = \begin{Bmatrix} \cos\delta_\infty \cos\alpha_\infty \\ \cos\delta_\infty \sin\alpha_\infty \\ \sin\delta_\infty \end{Bmatrix}$$

where

$$\alpha_\infty = \text{ right ascension of departure asymptote}$$

$$\delta_\infty = \text{ declination of departure asymptote}$$

The specific orbital energy is computed from the following expression

$$C_3 = v^2 - \frac{2\mu}{r}$$

The outgoing unit asymptote vector is given by

$$\hat{\mathbf{S}} = \frac{1}{1 + C_3 \dfrac{h^2}{\mu^2}} \left\{ \left(\frac{\sqrt{C_3}}{\mu} \right) \mathbf{h} \times \mathbf{e} - \mathbf{e} \right\} = \frac{1}{1 + C_3 \dfrac{p}{\mu}} \left\{ \left(\frac{\sqrt{C_3}}{\mu} \right) \mathbf{h} \times \mathbf{e} - \mathbf{e} \right\}$$

Finally, the targeting or constraint vector consists of the "energy-scaled" unit asymptote vector given by

$$\mathbf{S}_s = C_3 \hat{\mathbf{S}}$$

which is usually evaluated at the end of the final finite-burn orbit transfer maneuver or perhaps at a trajectory interface point (TIP) some time after final stage burnout.

References

"On the Equinoctial Orbital Elements", R. A. Brouke and P. J. Cefola, *Celestial Mechanics*, Vol. 5, pp. 303-310, 1972.

"A Set of Modified Equinoctial Orbital Elements", M. J. H. Walker, B. Ireland and J. Owens, *Celestial Mechanics*, Vol. 36, pp. 409-419, 1985.

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"Gamma Guidance for the Inertial Upper Stage (IUS)", John W. Hardtla, AIAA 78-1292, AIAA Guidance and Control Conference, Palo Alto, Calif., August 7-9, 1978.