ECI Equations of Vehicle Motion

The first-order, Earth-centered-inertial (ECI) equations of aerospace vehicle motion subject to gravity, aerodynamic forces and propulsive thrust are given by the following nonlinear system of vector differential equations:

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}$$

$$\frac{d\mathbf{v}}{dt} = \mathbf{a}_{grav} + \mathbf{a}_{aero} + \mathbf{a}_{thrust}$$

$$\dot{m} = -\frac{\eta T}{g\, I_{sp}}$$

where $\mathbf{r}$ is the ECI position vector of the vehicle, $\mathbf{v}$ is the ECI velocity vector, $\mathbf{a}_{grav}$ is the acceleration due to gravity, $\mathbf{a}_{aero}$ is the acceleration due to aerodynamic forces, $\mathbf{a}_{thrust}$ is the acceleration due to propulsive thrust, T is the propulsive thrust, η is the throttle setting, g is the acceleration of gravity at the Earth's surface and I_{sp} is the specific impulse of the propulsive system.

Gravity acceleration vector

The ECI components of the gravity acceleration including the effect of oblateness due to J_2 are given by

$$a_x = -\mu \frac{r_x}{r^3}\left\{1 + \frac{3}{2}\frac{J_2 r_{eq}^2}{r^2}\left(1 - \frac{5r_z^2}{r^2}\right)\right\}$$

$$a_y = -\mu \frac{r_y}{r^3}\left\{1 + \frac{3}{2}\frac{J_2 r_{eq}^2}{r^2}\left(1 - \frac{5r_z^2}{r^2}\right)\right\}$$

$$a_z = -\mu \frac{r_z}{r^3}\left\{1 + \frac{3}{2}\frac{J_2 r_{eq}^2}{r^2}\left(3 - \frac{5r_z^2}{r^2}\right)\right\}$$

where $r = \sqrt{r_x^2 + r_y^2 + r_z^2}$, and r_x, r_y and r_z are the rectangular components of the ECI position vector. In these equations μ and r_{eq} are the gravitational constant and equatorial radius of the Earth, respectively. For *unperturbed* or spherical gravity motion the components of the ECI gravity acceleration vector are simply

$$a_x = -\mu \frac{r_x}{r^3}$$

$$a_y = -\mu \frac{r_y}{r^3}$$

$$a_z = -\mu \frac{r_z}{r^3}$$

Aerodynamic acceleration vector

The ECI acceleration vector due to aerodynamic forces is given by

$$\mathbf{a}_{aero} = \left(\frac{D}{m}\right)\hat{\mathbf{i}}_D + \left(\frac{L}{m}\right)\hat{\mathbf{i}}_L = \mathbf{Q}\begin{bmatrix} -D/m \\ L\sin\beta/m \\ -L\cos\beta/m \end{bmatrix} = \left(\frac{1}{2m}\rho v_r^2 A\right)\mathbf{Q}\begin{bmatrix} -C_D \\ C_L \sin\beta \\ -C_L\cos\beta \end{bmatrix}$$

where

$D =$ aerodynamic drag force $= \frac{1}{2}\rho v_r^2 C_D A$

$L =$ aerodynamic lift force $= \frac{1}{2}\rho v_r^2 C_L A$

$\rho =$ atmospheric density

$v_r =$ relative velocity

$C_D =$ drag coefficient

$C_L =$ lift coefficient

$A =$ aerodynamic reference area

$m =$ vehicle mass

$\beta =$ bank angle

$\hat{\mathbf{i}}_D =$ ECI unit vector in the drag direction

$\hat{\mathbf{i}}_L =$ ECI unit vector in the lift direction

The transformation matrix $\mathbf{Q}$ can be determined from the ECI position vector $\mathbf{r}$ and the relative velocity vector $\mathbf{v}_r$ as follows

$$\mathbf{Q} = \left[\frac{\mathbf{v}_r}{|\mathbf{v}_r|},\quad \left(\frac{\mathbf{v}_r}{|\mathbf{v}_r|} \times \frac{\mathbf{r}}{|\mathbf{r}|}\right),\quad \frac{\mathbf{v}_r}{|\mathbf{v}_r|} \times \left(\frac{\mathbf{v}_r}{|\mathbf{v}_r|} \times \frac{\mathbf{r}}{|\mathbf{r}|}\right)\right]$$

The relative velocity vector $\mathbf{v}_r$ can be computed from

$$\mathbf{v}_r = \mathbf{v} - (\boldsymbol{\omega}_e \times \mathbf{r})$$

where $\boldsymbol{\omega}_e$ is the angular velocity vector of the Earth. This vector is equal to $\omega_e \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ where ω_e is the scalar inertial rotation rate of the Earth.

The expressions for the ECI unit vectors $\hat{\mathbf{i}}_D$ and $\hat{\mathbf{i}}_L$ are

$$\hat{\mathbf{i}}_D = -\frac{\mathbf{v}_r}{|\mathbf{v}_r|} = -\hat{\mathbf{v}}_r$$

$$\hat{\mathbf{i}}_L = \sin\beta\,\hat{\mathbf{i}}_y + \cos\beta\left(\hat{\mathbf{i}}_D \times \hat{\mathbf{i}}_y\right)$$

where $\hat{\mathbf{i}}_y = \hat{\mathbf{r}} \times \hat{\mathbf{i}}_D$, β is the bank angle of the vehicle and $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$ is the ECI unit position vector of the vehicle.

The lift and drag coefficients are typically functions of the angle-of-attack of the vehicle α and perhaps Mach number, dynamic pressure or altitude.

Thrust acceleration vector

The ECI acceleration vector due to a propulsive thrust force T is given by

$$\mathbf{a}_{thrust} = \mathbf{Q}\begin{bmatrix} T\cos\alpha/m \\ T\sin\alpha\sin\beta/m \\ -T\sin\alpha\cos\beta/m \end{bmatrix} = \mathbf{Q}\left(\frac{T_{vac} - A_e p}{m}\right)\begin{bmatrix} \cos\alpha \\ \sin\alpha\sin\beta \\ -\sin\alpha\cos\beta \end{bmatrix} = \frac{T}{m}\mathbf{u}_T$$

where T_{vac} is the vacuum thrust of the propulsion system, A_e is the nozzle exit area, p is the ambient atmospheric pressure and $\mathbf{u}_T$ is the ECI unit thrust vector.

Motion Relative to a Non-rotating Earth

Vehicle motion relative to a non-rotating Earth is modeled by substituting the inertial velocity vector $\mathbf{v}$ for the relative velocity vector $\mathbf{v}_r$ in the equations for the $\mathbf{Q}$ matrix, the unit vectors $\hat{\mathbf{i}}_D$ and $\hat{\mathbf{i}}_L$, and the equations for aerodynamics forces.